Chapter 13 - Two-Way Analysis of Variance

Statistics 104

Autumn 2004

Two-Way Analysis of Variance

Want to describe a continuous response variable with two categorical factors

Example 1: Kenton Food Example

$y =$ cases of cereal sold

Factor A: Colour (3 or 5) Factor B: Carton (Yes or No)

Example 2: Treating Toxic Agents

A study was part of an investigation into combating toxic agents. 3 poisons and 4 treatments, leading to 12 combinations were of interest. Each combination was studied on $n_{ij} = 4$ animals ($N = 48$ total observations).

$y =$ Survival time

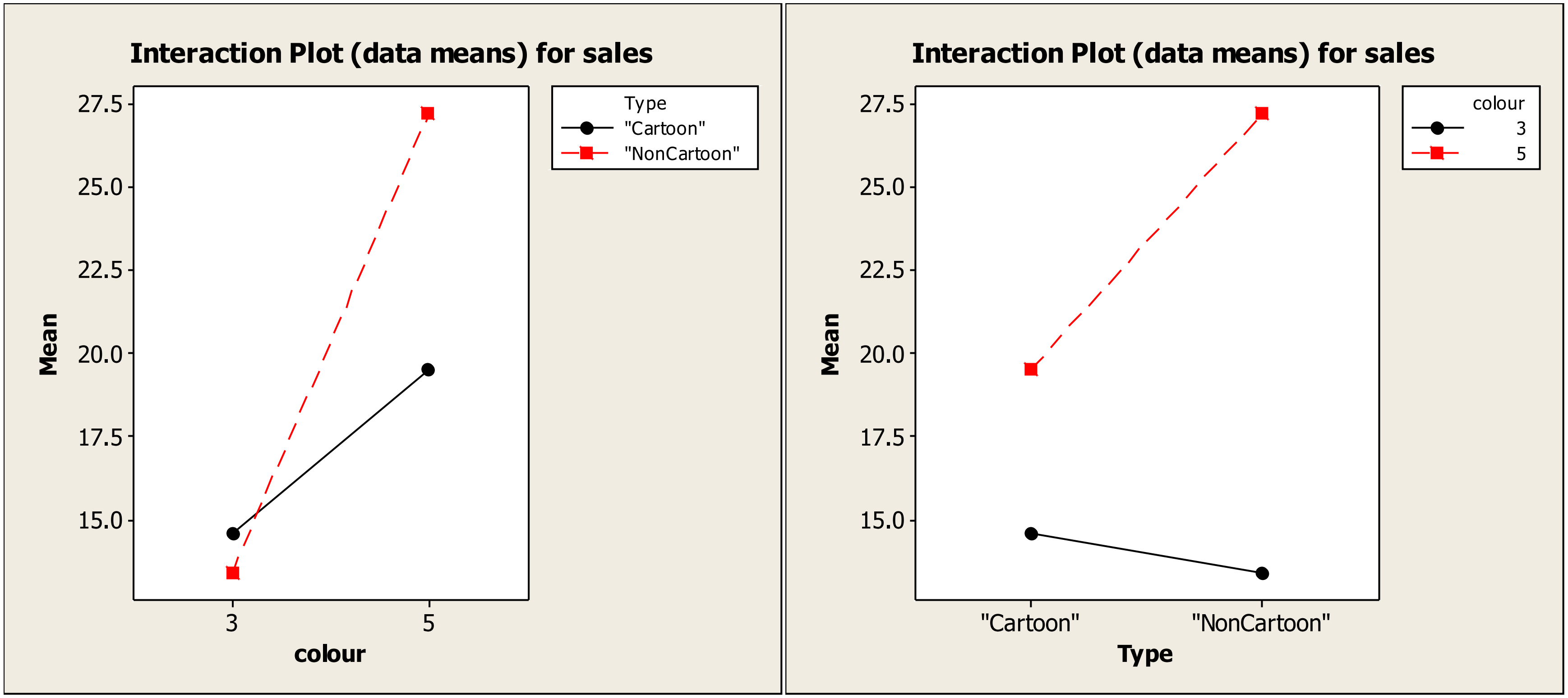
Factor A: Poison (I, II, III) Factor B: Treatment (A, B, C, D)

Advantages of two-way ANOVA (and higher way)

- Can study more than one factor at a time, potentially saving resources.
- Can reduce residual variation by including a second factor thought to influence the response.
- Can investigate interactions

Interaction: The effect of changing the level of one predictor variable depends on the level of another predictor variable.

The following plots, sometimes referred to as an interaction plot, plots the average for each combination of factors. One factor is used for the levels of the x-axis and the averages are joined based on the second factor.

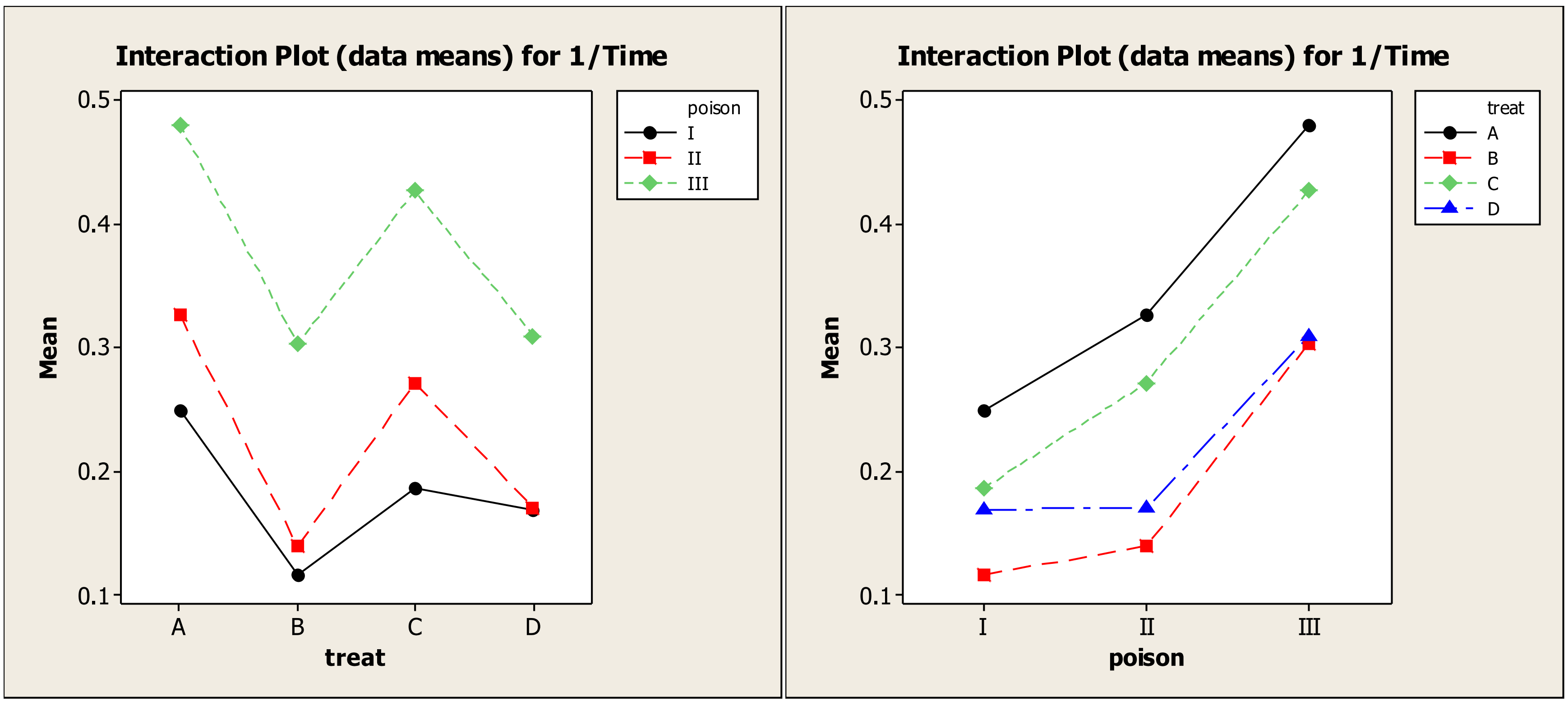


The effect of switching from a 3 to 5 colour design is different for cartoon and non-cartoon designs.

Or you can look at it as the effect of switching from a cartoon design to a non-cartoon design appears to be different for 3 and 5 colours.

Which factor to use on the x-axis often doesn't matter. Find the one which displays the features of the data better.

The toxic agents example is a situation where there doesn't appear to be an interaction. A lack of an interaction is suggested by roughly parallel times.



Two-Way ANOVA Model

$$y_{ij} = \mu_{ij} + \epsilon_{ijk}; \quad \epsilon_{ijk} \sim N(0, \sigma)$$

i: level of factor A (I levels)

j: level of factor B (J levels)

k: observation within i & j combination (n_{ij} observations)

$$y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

μ: overall mean effect

α_i: A main effects

β_j: B main effects

$(\alpha\beta)_{ij}$: AB interaction effects

Fitting the Model:

The treatment effects are estimated by

$$\bar{y}_{ij} = \frac{1}{n_{ij}} \sum_k y_{ijk}$$

The standard deviation of the errors is estimated by the pooled procedure again

$$\begin{aligned} s_p^2 &= \frac{\sum (n_{ij} - 1) s_{ij}^2}{N - IJ} \\ s_p &= \sqrt{s_p^2} \end{aligned}$$

Decomposition of Effects:

As with one-way ANOVA, the variation in the response variable can be broken down into different terms

At the initial level, it is the same as for the one-way model

$$SST = SSM + SSE \qquad DFT = DFM + DFE$$

However the Model SS and DF can be broken up into

$$\begin{aligned} SSM &= SSA + SSB + SSAB \\ DFM &= DFA + DFB + DFAB \end{aligned}$$

- SSA represents the variation of the means for the different levels of factor A (A main effect)

- SSB represents the variation of the means for the different levels of factor B (B main effect)

- SSAB represents the additional variation of the means described by the interaction effect (AB interaction)

$$SSAB = SSM - SSA - SSB$$

The degrees of freedom for the model has a similar decomposition

$$\begin{aligned} DFA &= I - 1 \\ DFB &= J - 1 \\ DFAB &= DFM - DFA - DFB \\ &= (IJ - 1) - (I - 1) - (J - 1) = (I - 1)(J - 1) \end{aligned}$$

Inference for Two-way ANOVA

Based on the sums of squares decomposition.

ANOVA Table:

Source	DF	SS	MS	F
A	$I-1$	SSA	$MSA = \frac{SSA}{DFA}$	$F = \frac{MSA}{MSE}$
B	$J-1$	SSB	$MSB = \frac{SSB}{DFB}$	$F = \frac{MSB}{MSE}$
AB	$(I-1)(J-1)$	$SSAB$	$MSAB = \frac{SSAB}{DFAB}$	$F = \frac{MSAB}{MSE}$
Error	$N-IJ$	SSE	$MSE = \frac{SSE}{DFE}$	
Total	$N-1$	SST		

There are three hypotheses that can be investigated in a two-way ANOVA, the A main effect, the B main effect, and the AB interaction.

(Note: unless n_{ij} are the same for all i & j combinations, the hypotheses being examined can change. For more information, see a more advanced design texts such as Dean and Voss or Montgomery.)

The significance of each of the effects can be examined with the three F tests.

- A main effect:

$$F_A = \frac{MSA}{MSE}$$

- B main effect:

$$F_B = \frac{MSB}{MSE}$$

- AB interaction:

$$F_{AB} = \frac{MSAB}{MSE}$$

Each of these observed F statistics is compared to an F distribution with the degrees of freedom given by the two terms in the ratio (e.g. DFAB, DFE for the interaction test).

The p-value for tests are given by

$$p\text{−value} = P[F \geq F_{obs}]$$

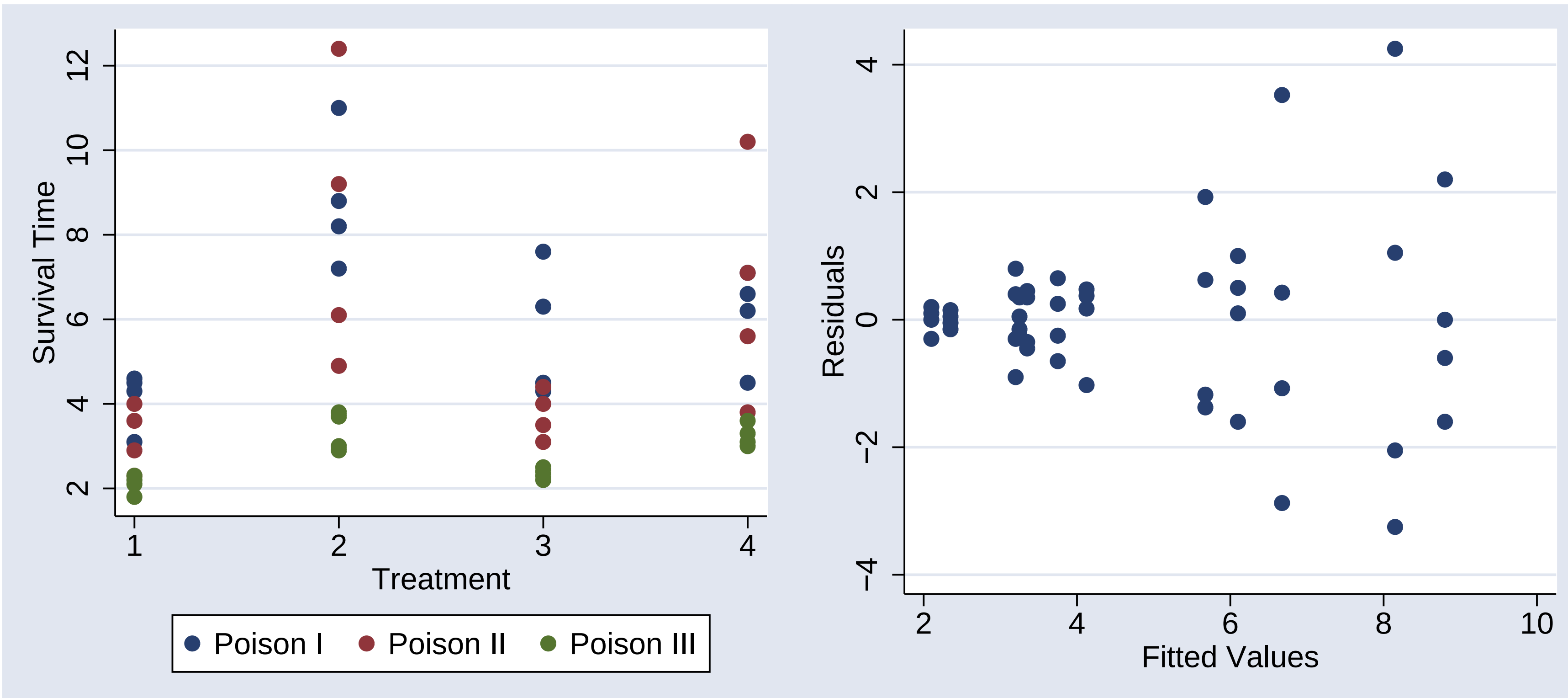
Normally you start with the interaction test first, as if there is a significant interaction, it can influence the interpretation of the main effects. It can also affect the other hypothesis tests if the design isn't balanced (same number of observations on each factor combination).

Also if the interaction is important, it implies that both variables are important and that you need to know the level of both variables to describe how the response might change.

Often when the interaction is significant, the main effects won't even be examined.

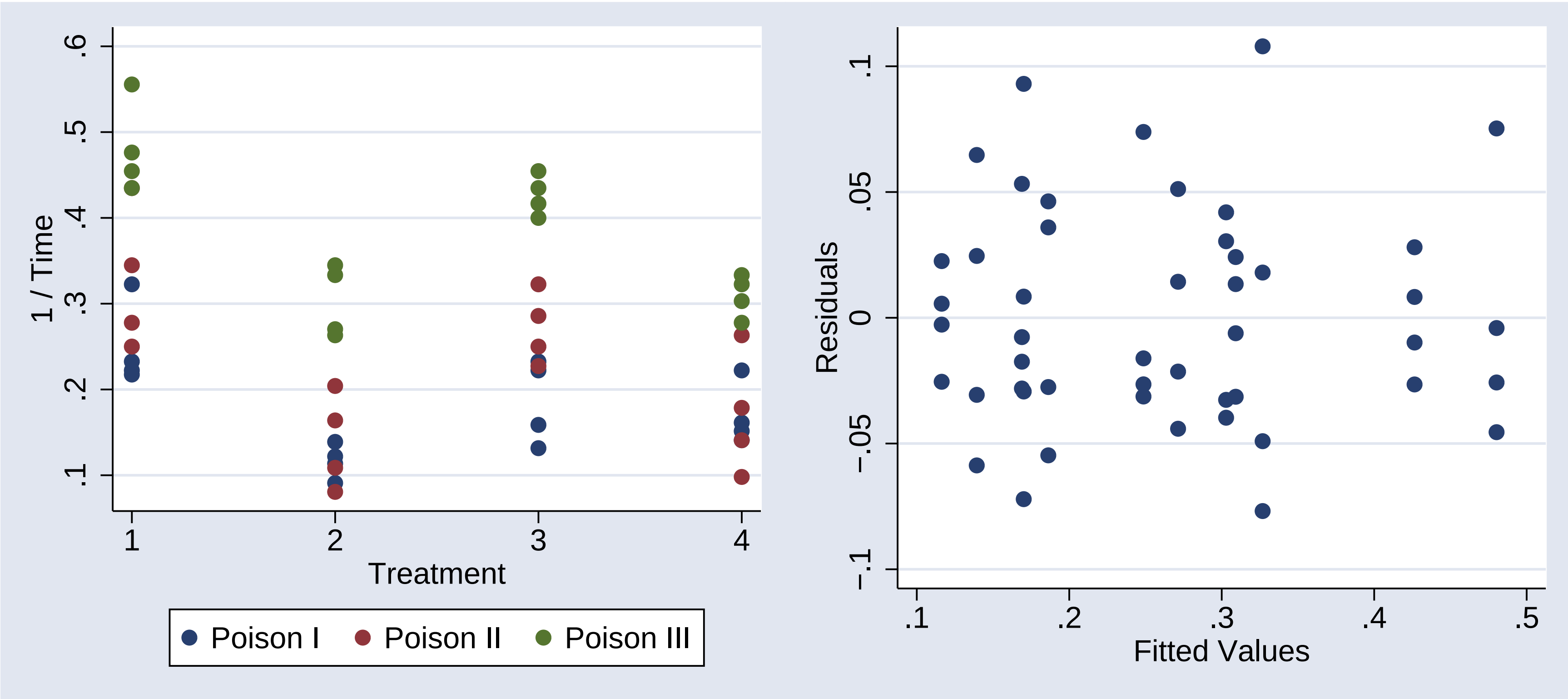
Example: Toxic Agents

Instead of analyzing the survival times, instead we will analyze 1/Times since the survival time data doesn't satisfy the constant variance assumption.



This data shows an increasing variance with large survival times.

The transformation $\frac{1}{Time}$ is one possible way to deal with this. Often looking at $\frac{1}{Time}$ makes sense as well, as it converts times to rates.



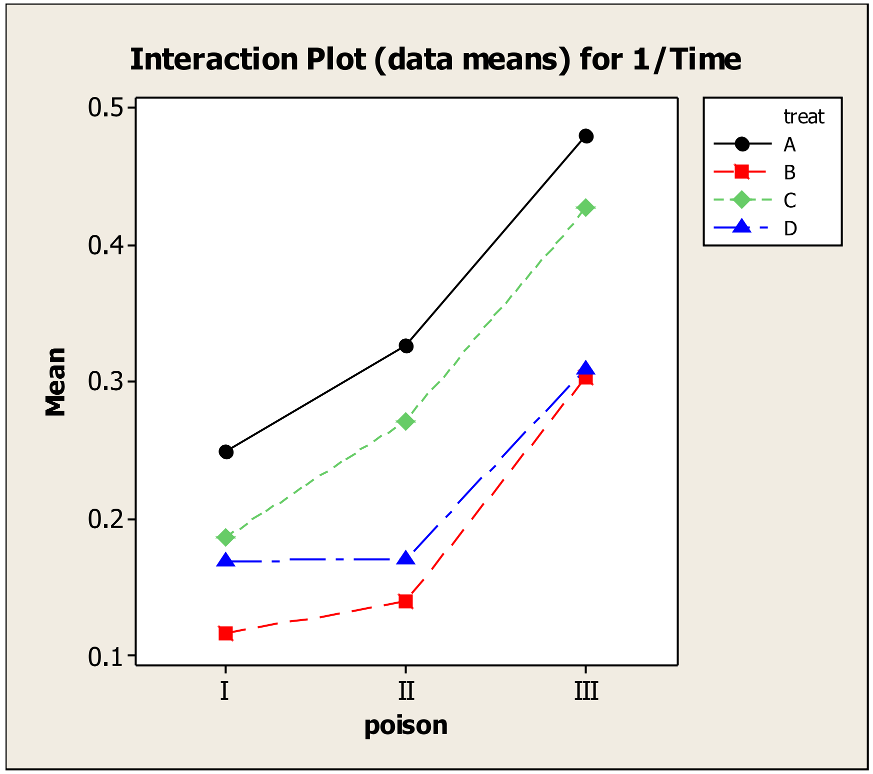
```
. anova rate poison treat poison*treat, partial

                  Number of obs =      48     R-squared     =  0.8681
                  Root MSE      = .048999     Adj R-squared =  0.8277

      Source |  Partial SS    df       MS           F     Prob > F
-------------+----------------------------------------------------
       Model |  .568621825    11  .051692893      21.53     0.0000
             |
      poison |  .348771201     2    .1743856      72.63     0.0000
       treat |    .2041429     3  .068047633      28.34     0.0000
poison*treat |  .015707724     6  .002617954       1.09     0.3867
             |
    Residual |  .086430836    36  .002400857
-------------+----------------------------------------------------
       Total |  .655052661    47  .013937291
```

The test for the interaction in this example is not significant.

However both main effects are significant.



From examining this interaction plot, it appears that treatment A has the fastest death rate (its the the top line) and treatments B and D have the slowest death rates.

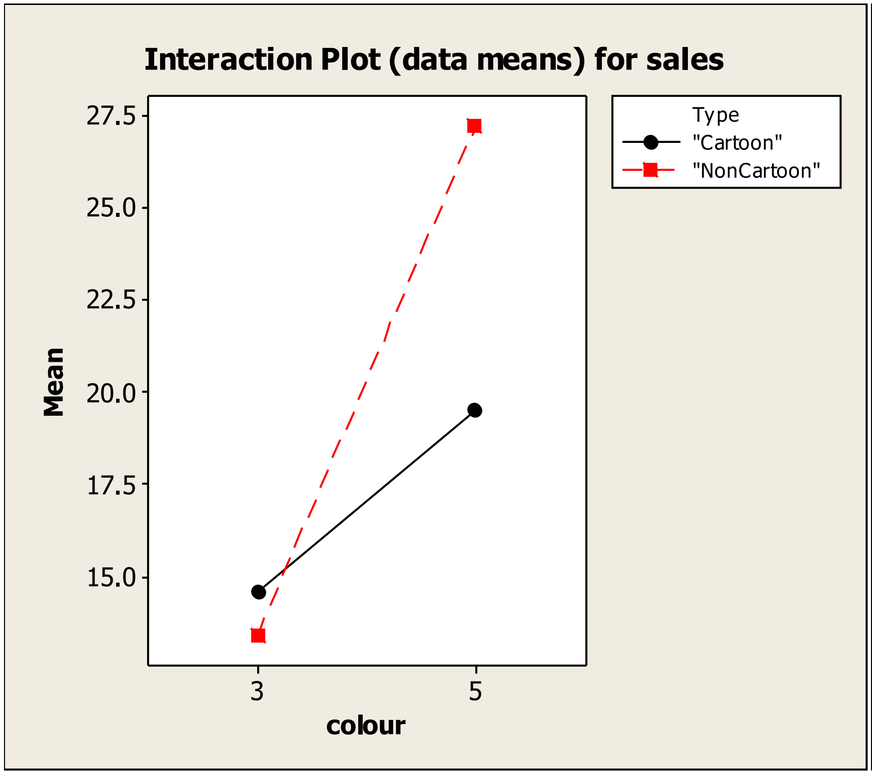
Poison III seems to be the most deadly (highest rate for each treatment).

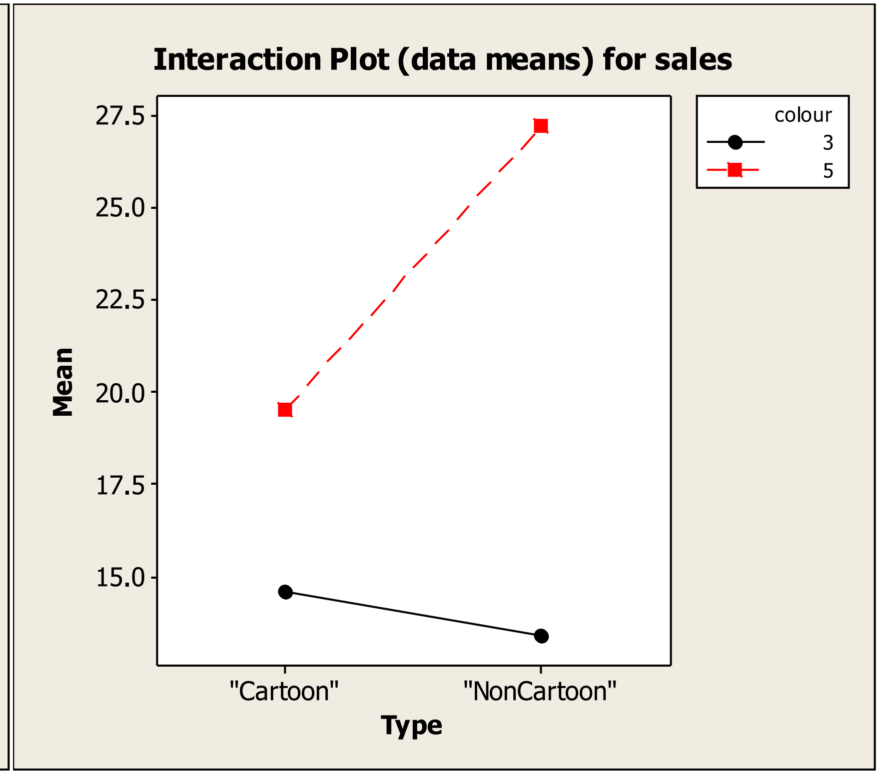
Example: Kenton Sales data

```
. anova sales colour Type colour*Type, sequential

                   Number of obs =      19     R-squared     =  0.7881
                   Root MSE      = 3.24756     Adj R-squared =  0.7457

          Source |    Seq. SS     df       MS           F     Prob > F
     ------------+----------------------------------------------------
           Model |  588.221053     3  196.073684      18.59     0.0000
                 |
          colour |  452.865497     1  452.865497      42.94     0.0000
            Type |  42.1673203     1  42.1673203       4.00     0.0640
     colour*Type |  93.1882353     1  93.1882353       8.84     0.0095
                 |
        Residual |       158.2    15  10.5466667
     ------------+----------------------------------------------------
           Total |  746.421053    18  41.4678363
```





For this example, the interaction term is significant (p-value $= 0.0095$). So to determine which combination will lead to optimal sales you need to look at the combination of the two factors.

It appears in this case to be the non-cartoon, 5 colour design.

Note that the following analysis is reasonable, since the interaction model can be made equivalent to a one-way ANOVA model. (It ignores the structure of the treatment combinations.)

```
. oneway sales design, bonferroni tabulate

                  |          Summary of Sales
           Design |        Mean   Std. Dev.       Freq.
------------------+------------------------------------
(3 cartoon)     1 |        14.6   2.3021729           5
(3 non-cartoon) 2 |        13.4   3.6469165           5
(5 cartoon)     3 |        19.5   2.6457513           4
(5 non-cartoon) 4 |        27.2   3.9623226           5
------------------+------------------------------------
            Total |   18.631579   6.4395525          19
```

```
                    Comparison of Sales by Design
                            (Bonferroni)
Row Mean-|
Col Mean |          1          2          3
---------+---------------------------------
       2 |       -1.2
         |      1.000
         |
       3 |        4.9        6.1
         |      0.240      0.081
         |
       4 |       12.6       13.8        7.7
         |      0.000      0.000      0.018
```

All the tests adjusting for the multiple comparisons by the Bonferroni procedure involving treatment 4 (5 colour, non-cartoon) are significant, suggesting that this packaging is preferable, since the estimated difference is positive.

A better comparison procedure would be to look at Tukey based confidence intervals for the differences as they give smaller confidence intervals than Bonferroni does.

```
. prcomp sales design, tukey

                   Pairwise Comparisons of Means

Response variable (Y): sales       Sales
Group variable (X):    design      Design

  Group variable (X): design        Response variable (Y): sales
-------------------------------    -------------------------------
      Level                              n          Mean         S.E.
------------------------------------------------------------------
          1                              5          14.6     1.029563
          2                              5          13.4     1.630951
          3                              4          19.5     1.322876
          4                              5          27.2     1.772005
------------------------------------------------------------------
```

```
Simultaneous confidence level: 95%    (Tukey wsd method)
Homogeneous error SD = 3.247563, degrees of freedom = 15
```

Level(X)	Mean(Y)	Level(X)	Mean(Y)	Diff Mean	95% Confidence Limits	
2	13.4	1	14.6	-1.2	-7.119742	4.719742
3	19.5	1	14.6	4.9	-1.378834	11.17883
		2	13.4	6.1	-.1788342	12.37883
4	27.2	1	14.6	12.6	6.680258	18.51974
		2	13.4	13.8	7.880258	19.71974
		3	19.5	7.7	1.421166	13.97883

All the intervals involving treatment 4 (5 colour, non-cartoon) are strictly positive, supporting that the expected sales on this combination are higher than the other 3 packages.