Section 4.5 - General Probability Rules

Statistics 104

Autumn 2004

General Probability Rules

Rules of Probability - So Far

1. $0 \leq P[A] \leq 1$ for any event A.

2. $P[S] = 1$.

3. **Complement Rule**: For any event A,

$$P[A^c] = 1 - P[A]$$

4. **Addition Rule**: If A and B are **disjoint** events,

$$P[A \text{ or } B] = P[A] + P[B]$$

5. **Multiplication Rule**: If A and B are **independent** events,

$$P[A \text{ and } B] = P[A] \times P[B]$$

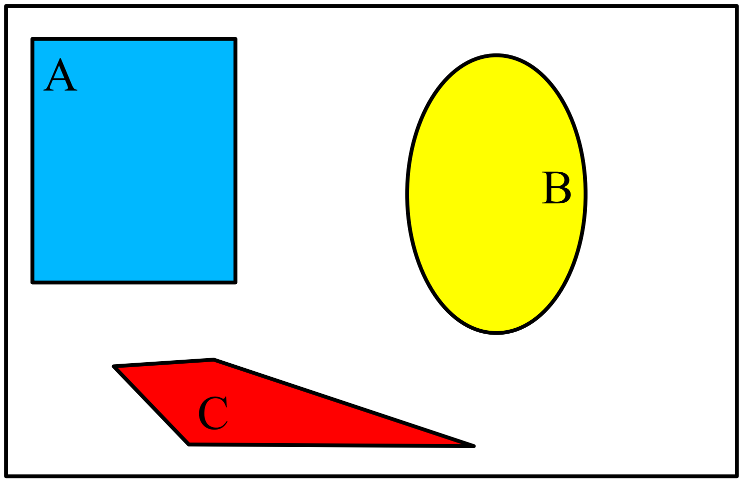
For these last two rules there are restrictions that we have seen matter. How can we modify the rules to make them more general.

Addition Rule for Disjoint Events

If A, B, and C are disjoint events, then

$$P[A \text{ or } B \text{ or } C] = P[A] + P[B] + P[C]$$

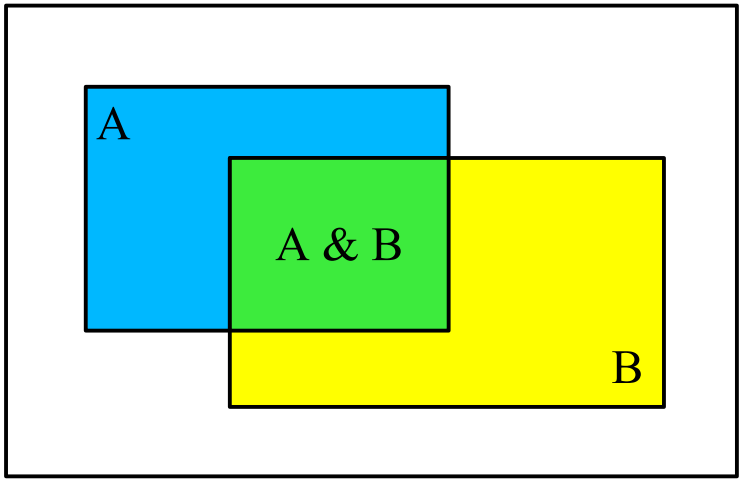
This rule extends to an arbitrary number of disjoint events.



General Rule for Unions of 2 Events

Note that the two events A & B do not need to be disjoint. As we've seen earlier, adding the two probabilities will not give the correct answer for $P[A \text{ or } B]$. Instead,

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$



$$P[A] = P[A \text{ and } B] + P[A \text{ and } B^c]$$

$$P[B] = P[A \text{ and } B] + P[A^c \text{ and } B]$$

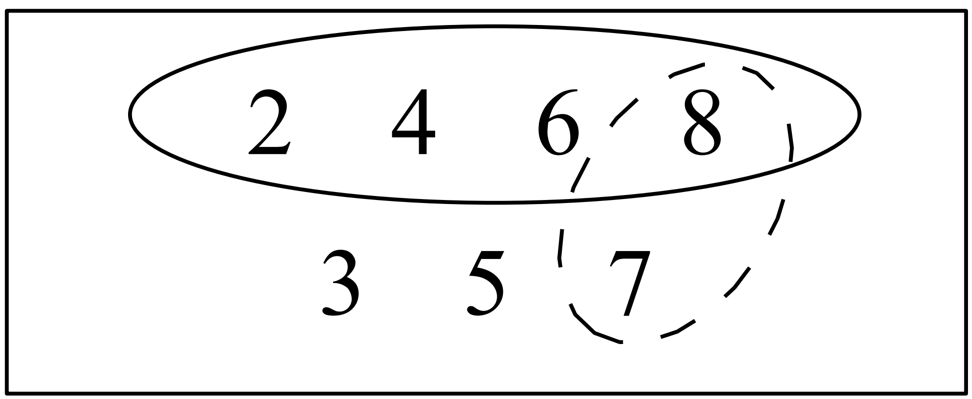
The event A and B gets double counted in $P[A] + P[B]$.

Note that the disjoint rule is a special case of this, since for disjoint events A and B, $P[A \text{ and } B] = 0$.

Example: Sum of two rolls of a 4-sided die

- $A =$ Sum is even $= \{2, 4, 6, 8\}$
- $C =$ Sum $> 6 = \{7, 8\}$

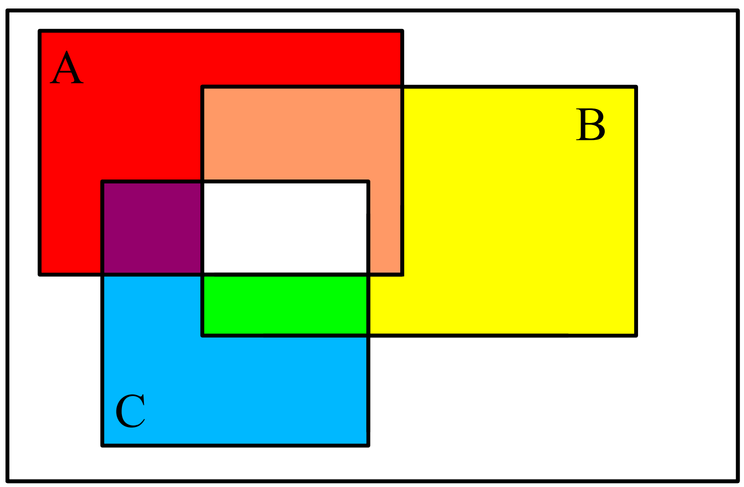
$$P[A \text{ or } C] = P[A] + P[C] - P[A \text{ and } C] = \frac{8}{16} + \frac{3}{16} - \frac{1}{16} = \frac{10}{16}$$



More complicated versions of this rule exist for 3 or more events.

For example, for three events A, B, and C,

$$
\begin{aligned}
P[A \text{ or } B \text{ or } C] = P[A] + P[B] + P[C] \\
-P[A \text{ and } B] - P[A \text{ and } C] - P[B \text{ and } C] \\
+P[A \text{ and } B \text{ and } C]
\end{aligned}
$$



Conditional Probability

Interested in two events A and B. Suppose that we know that A occurs. What is the probability that B occurs knowing that A occurs?

Examples:

- Rainfall

  A: Rains today
  B: Rains tomorrow

  Does knowing whether it rains today change our belief that it will rain tomorrow.

  Denoted $P[B|A]$ (the probability of B given A occurs).

- Switches

  Does knowing whether the switch is from company 1 or 2 tell us anything about whether the switch is defective?

- Disease testing

  A: has disease
  B: positive test

  $P[+ \text{ test} \mid \text{has disease}] = 0.98$

  98% of the time, tests on people with the disease come up positive. ($P[B|A]$)

  $P[+ \text{ test} \mid \text{no disease}] = 0.07$ ($P[B|A^c]$)

  These imply that

  $P[- \text{ test} \mid \text{has disease}] = 1 - 0.98 = 0.02$

  $P[- \text{ test} \mid \text{no disease}] = 1 - 0.07 = 0.93$

  What is the probability of

$$P[+ \text{ test} \ \& \text{ has disease}]$$

  or

$$P[- \text{ test} \ \& \text{ no disease}]$$

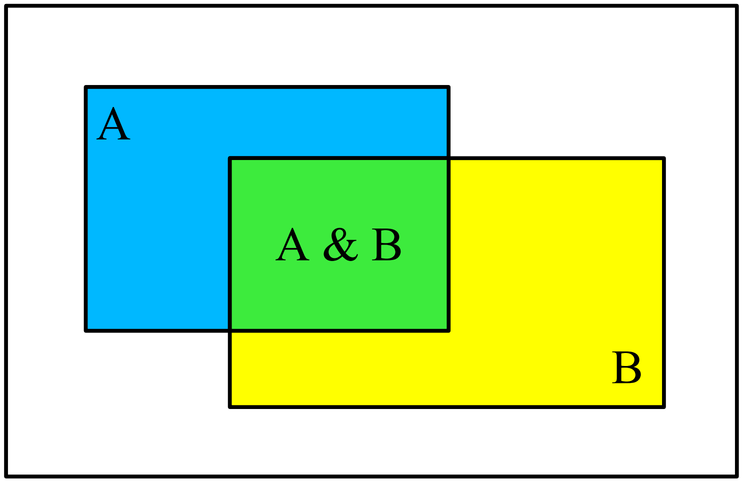
To answer these questions, we also need to know

$$P[\text{disease}] = 0.01 \quad (P[A]) \quad \Rightarrow \quad P[\text{no disease}] = 0.99 \quad (P[A^c])$$

General Multiplication Rule

$$P[A \text{ and } B] = P[A] \times P[B|A]$$

To be in both, first you must be in A (giving the $P[A]$ piece), then given that you are in A, you must be in B (giving the $P[B|A]$ piece).



$$\begin{aligned} P[+\text{ test \& has disease}] &= P[\text{has disease}]P[+\text{ test} \mid \text{has disease}] \\ &= 0.01 \times 0.98 = 0.0098 \end{aligned}$$

$$\begin{aligned} P[-\text{ test \& no disease}] &= P[\text{no disease}]P[-\text{ test} \mid \text{no disease}] \\ &= 0.99 \times 0.93 = 0.9207 \end{aligned}$$

	Disease	No Disease	Test Status
+ Test	$0.01 \times 0.98 = 0.0098$	$0.99 \times 0.07 = 0.0693$	0.0791
– Test	$0.01 \times 0.02 = 0.0002$	$0.99 \times 0.93 = 0.9207$	0.9209
Disease Status	0.01	0.99	1

Marginal Probabilities

$P[+\text{ test}]$ and $P[-\text{ test}]$

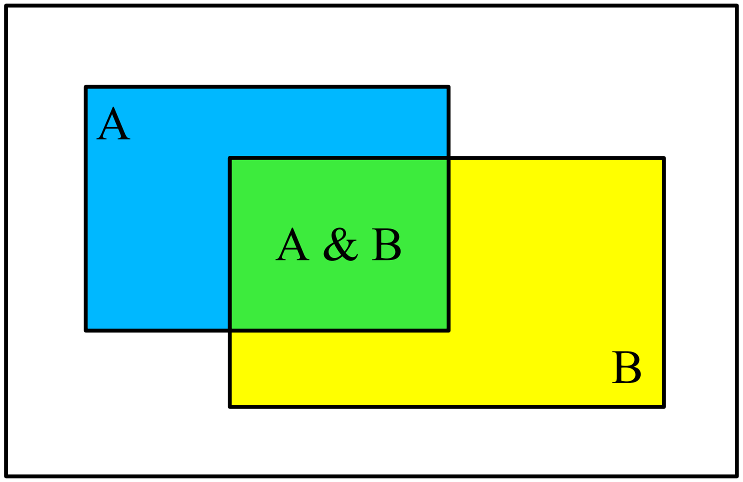
$$\begin{aligned} P[+\text{ test}] &= P[+\text{ test \& has disease}] + P[+\text{ test \& no disease}] \\ &= 0.0098 + 0.0693 = 0.0791 \end{aligned}$$

$$\begin{aligned} P[-\text{ test}] &= P[-\text{ test \& has disease}] + P[-\text{ test \& no disease}] \\ &= 0.0002 + 0.9207 = 0.9209 \\ &= 1 - 0.0791 \end{aligned}$$

These were gotten by adding across each row of the table.

The probabilities discussed in the example are based on an ELISA (enzyme-linked immunosorbent assay) test for HIV.

Conditional Probability



When $P[A] > 0$

$$P[B|A] = \frac{P[A \text{ and } B]}{P[A]}$$

What is $P[\text{disease} \mid + \ \text{test}]$ and $P[\text{disease} \mid + \ \text{test}]$?

$$\begin{aligned} P[\text{disease} \mid + \ \text{test}] &= \frac{P[+ \text{ test \& has disease}]}{P[+ \text{ test}]} \\ &= \frac{0.0098}{0.0791} = 0.124 \end{aligned}$$

$$
\begin{aligned}
P[\text{ no disease } | - \text{ test}] &= \frac{P[-\text{ test \& no disease}]}{P[-\text{ test}]} \\
&= \frac{0.9207}{0.9209} = 0.99978
\end{aligned}
$$

Note that $P[A|B]$ and $P[B|A]$ are two completely different quantities.

Note that the other rules of probability must be satisfied by conditional probabilities. For example $P[B^c|A] = 1 - P[B|A]$.

Bayes' Rule

Bayes' Rule gives an approach to calculating conditional probabilities. It allows for switching the order of conditioning (from $P[B|A]$ to $P[A|B]$).

$$
\begin{aligned}
P[A|B] &= \frac{P[B|A]P[A]}{P[B|A]P[A] + P[B|A^c]P[A^c]} \\
&= \frac{P[A\&B]}{P[A\&B] + P[A^c\&B]} = \frac{P[A\&B]}{P[B]}
\end{aligned}
$$

Assuming that $P[A]$ isn't 0 or 1.

Note that the above picture is the only known portrait of the Reverend Thomas Bayes F.R.S. (1701? - 1761), who derived the result. However there are questions to whether this is actually Bayes. The methods described could be used to determine $P[\text{Actual picture of Bayes} \mid \text{Information on picture and Bayes}]$.

The earlier calculations of $P[\text{disease} \mid + \ \text{test}]$ and $P[\text{ no disease} \mid - \ \text{test}]$ were applications of this rule. The different steps were broken down into their constituent pieces.

Example (Monty Hall problem):

There are three doors. One has a car behind it and the other two have farm animals behind them. You pick a door, then Monty will open another door and show you some farm animals and allow you to switch. You then win whatever is behind your final door choice.

You choose door 1 and then Monty opens door 2 and shows you the farm animals. Should you switch to door 3?

Answer: It depends

Three competing hypotheses D_1, D_2, and D_3 where

$$D_i = \{\text{Car is behind door } i\}$$

What should our observation A (the event we want to condition on) be?

We want to condition on all the available information, implying we should use

$A = \{$After door 1 was selected, Monty chose door 2 to be opened$\}$

Prior probabilities on car location: $P[D_i] = \frac{1}{3}, i = 1, 2, 3$

Likelihoods:

$$\begin{aligned} P[A|D_1] &= \frac{1}{2} \qquad\qquad (*) \\ P[A|D_2] &= 0 \\ P[A|D_3] &= 1 \end{aligned}$$

$$P[A] = \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{1}{2}$$

Conditional probabilities on car location:

$$\begin{aligned} P[D_1|A] &= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3} \qquad \text{No change!} \\ P[D_2|A] &= \frac{0 \times \frac{1}{3}}{\frac{1}{2}} = 0 \\ P[D_3|A] &= \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \qquad \text{Bigger!} \end{aligned}$$

If you are willing to assume that when two empty doors are available, Monty will randomly choose one of them to open (with equal probability) (assumption *), then you should switch. You'll win the car $\frac{2}{3}$ of the time.

Now instead, assume Monty opens the door based on the assumption

$$P[A|D_1] = 1 \qquad\qquad (**)$$

i.e. Monty will always choose door 2 when both doors 2 and 3 have animals behind them. (The other two are the same as before.) Now

$$P[A] = 1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$$

Now the posterior probabilities are

$$
\begin{aligned}
P[D_1|A] &= \frac{1 \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \\
P[D_2|A] &= 0 \\
P[D_3|A] &= \frac{1 \times \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}
\end{aligned}
$$

So in this situation, switching doesn't help (doesn't hurt either).

Note: This is an extremely subtle problem that people have discussed for years (go do a Google search on Monty Hall Problem). Some of the discussion goes back before the show Let Make a Deal ever showed up on TV. The solution depends on precise statements about how doors are chosen to be opened. Changing the assumptions can lead to situations that changing can't help and I believe there are situations where changing can actually be worse. They can also lead to situations where the are advantages to picking certain doors initially.

The General Multiplication Rule mentioned earlier can be extended. For example

$$P[A \text{ and } B \text{ and } C] = P[A]P[B|A]P[C|A \text{ and } B]$$

The key in applying this extension to more events is that the conditioning has to be all of the preceding events, i.e.

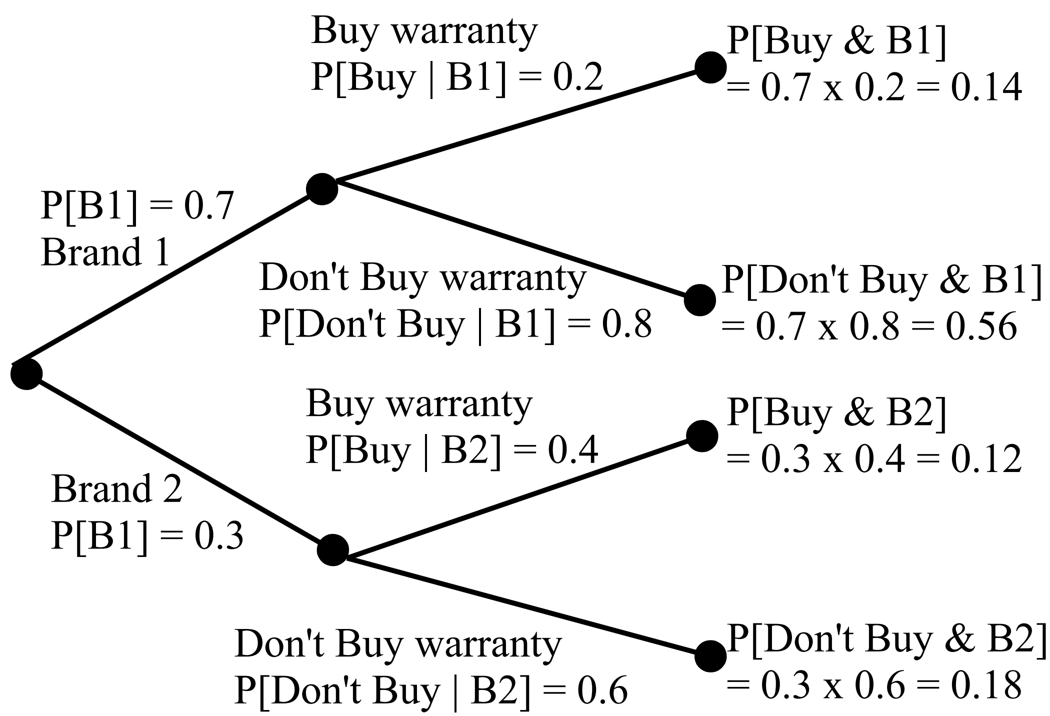
$$P[ABCD] = P[A]P[B|A]P[C|AB]P[D|ABC]$$

This approach can be used to build up extremely flexible models for modelling complex phenomena. (Hierarchical modelling)

This is precisely the approach taken in the Sea Surface Temperature example discussed on the first day of class.

Tree diagrams

Example: Warranty purchases



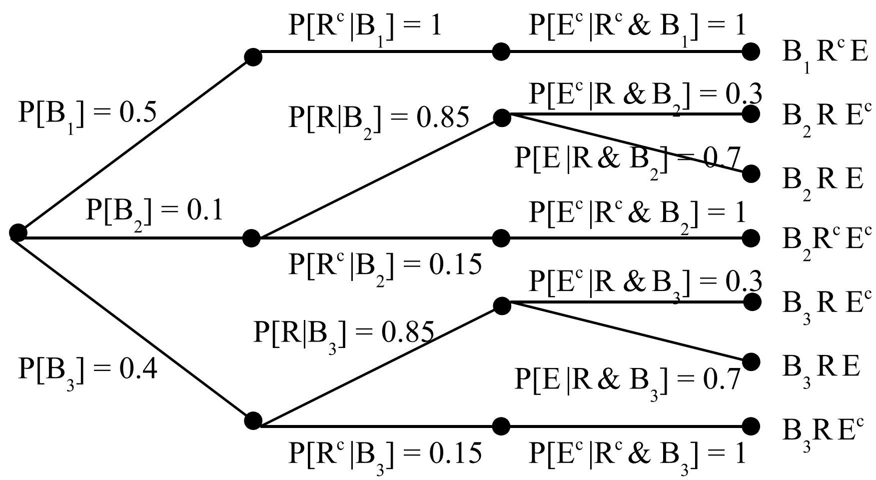
Each branch corresponds to a different event and each has a conditional probability associated with it.

By following the path and multiplying the probabilities along the path, the probability of any combination of events can be calculated.

Example: Jury selection

When a jury is selected for a trial, it is possible that a prospective can be excused from duty for cause. The following model describes a possible situation.

- Bias of randomly selected juror



- B_1: Unbiased
- B_2: Biased against the prosecution
- B_3: Biased against the defence

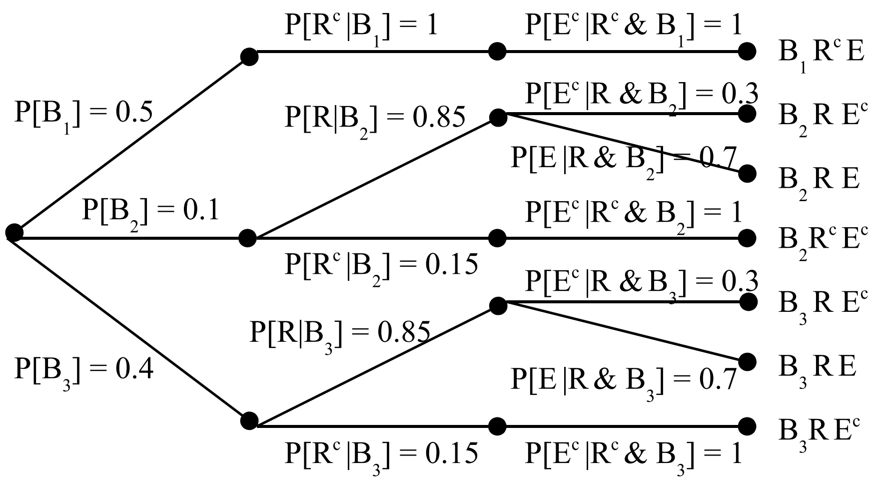
- R: Existing bias revealed during questioning

- E: Juror excused for cause

The probability of any combination of these three factors can be determined by multiplying the correct conditional probabilities. The probabilities for all twelve possibilities are

	R		R^c	
	E	E^c	E	E^c
B_1	0	0	0	0.5000
B_2	0.0595	0.0255	0	0.0150
B_3	0.2380	0.1020	0	0.0600

For example $P[B_2 \cap R \cap E^c] = 0.1 \times 0.85 \times 0.3 = 0.0255.$



From this we can get the following probabilities

$$P[E] = 0.2975 \quad P[E^c] = 0.7025$$
$$P[R] = 0.4250 \quad P[R^c] = 0.5750$$

$$\begin{array}{lll} P[B_1 \cap E] = 0 & P[B_2 \cap E] = 0.0595 & P[B_3 \cap E] = 0.238 \\ P[B_1 \cap E^c] = 0.5 & P[B_2 \cap E^c] = 0.0405 & P[B_3 \cap E^c] = 0.162 \end{array}$$

From these we can get the probabilities of bias status given that a person was not excused for cause from the jury.

$$P[B_1|E^c] = \frac{0.5}{0.7025} = 0.7117 \quad P[B_1|E] = \frac{0}{0.2975} = 0$$

$$P[B_2|E^c] = \frac{0.0405}{0.7025} = 0.0641 \quad P[B_2|E] = \frac{0.0595}{0.2975} = 0.2$$

$$P[B_3|E^c] = \frac{0.1620}{0.7025} = 0.2563 \quad P[B_3|E] = \frac{0.238}{0.2975} = 0.8$$

Independence and Conditional Probability

Two events A and B are independent if

$$P[B|A] = P[B]$$

So knowledge about whether A occurs or not, tells us nothing about whether B occurs.

Example: HIV test

- $P[\text{Disease}] = 0.01$
- $P[\text{Disease}| + \text{ test}] = 0.124$
- $P[\text{Disease}| - \text{ test}] = 0.00022$

So disease status and the test results are dependent.

Example: Switch example

Sample two switches from a batch and examine whether they meet design standards

	1st OK	1st Defective	
2nd OK	0.9025	0.0475	0.95
2nd Defective	0.0475	0.0025	0.05
	0.95	0.05	1

$P[\text{1st Defective}] = 0.05$

$P[\text{1st Defective}|\text{2nd Defective}] = 0.05$

So whether the 1st switch is defective is independent of whether the 2nd is defective.

So this gives another way of checking for independence in addition to the earlier idea of seeing whether $P[A \text{ and } B] = P[A]P[B]$.

The multiplication rule for independent events is just a special case of the more general multiplication rule.

If A and B are independent $P[A] = P[A|B]$, so

$$P[A|B]P[B] = P[A]P[B]$$

Also if $P[A|B] = P[A]$, then $P[B|A] = P[B]$. (Independence has no direction)