1. (15 points) Diane, Mike and Lew are employed by the Department of Public Health. One of their duties is being on "on call" during nonworking hours to handle any incidents that might endanger public safety. Each carries a beeper that can be activated if necessary. Diane is a conscientious worker and is within earshot of the beeper and can respond 90% of the time, while Mike is somewhat less reliable and can only respond 60% of the time, while Lew is unreliable as he will only respond 20% of the time. For each call, each of the three responds independently of the rest.

   (a) (5 points) What is the probability at least one of them responds?

$$\begin{aligned} P[\text{At least one responds}] &= 1 - P[\text{None respond}] \\ &= 1 - (1-0.9)(1-0.6)(1-0.2) = 0.968 \end{aligned}$$

   (b) (5 points) Given that Diane responds to a call, what is the probability that Lew also responds to this call?

   $P[\text{Lew responds}|\text{Diane responds}] = P[\text{Lew responds}] = 0.2$ since the events are independent.

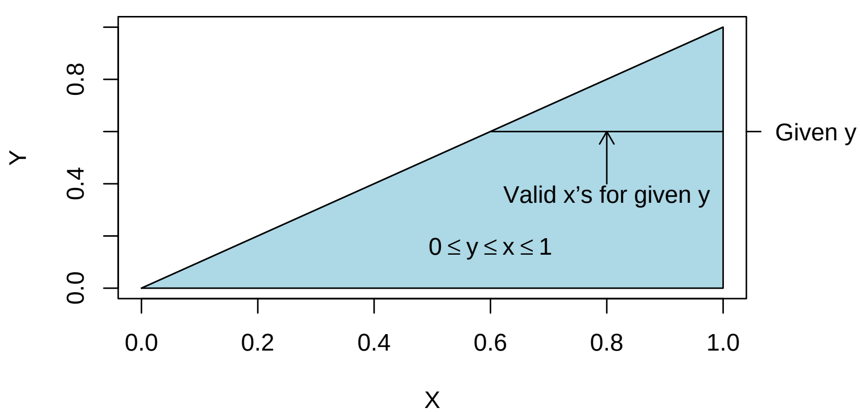
   (c) (5 points) What is the probability that Mike or Lew respond?

$$P[M \cup L] = P[M] + P[L] - P[M \cap L] = 0.6 + 0.2 - 0.6 \times 0.2 = 0.68$$

2. (20 points) Assume that 2 tasks are to be performed. Let X be the total time taken for the two tasks and Y be the time taken to complete just the first task (times recorded in hours). Assume that these two continuous random variables have a joint density

$$f_{X,Y}(x,y) = 8xy; \qquad 0 \le x \le 1, 0 \le y \le x$$

   (a) (5 points) Find $f_Y(y)$, the marginal density of the time to complete the first task.



$$
\begin{aligned}
f_Y(y) &= \int_y^1 8xy dx \\
&= 4y \int_y^1 2x dx \\
&= 4y\, x^2\Big|_y^1 \\
&= 4y(1-y^2); \qquad 0 \le y \le 1
\end{aligned}
$$

(b) (5 points) What are the mean and variance of X? You may assume that the marginal density of X is $f_X(x) = 4x^3;\ 0 \le x \le 1$.

$$E[X^n] = \int_0^1 x^n \times 4x^3 dx = \int_0^1 4x^{3+n} dx = \frac{4}{4+n}$$

so $E[X] = \frac{4}{5}$ and $E[X^2] = \frac{4}{6}$ which gives the variance of

$$\text{Var}(X) = \frac{4}{6} - \left(\frac{4}{5}\right)^2 = 0.02667$$

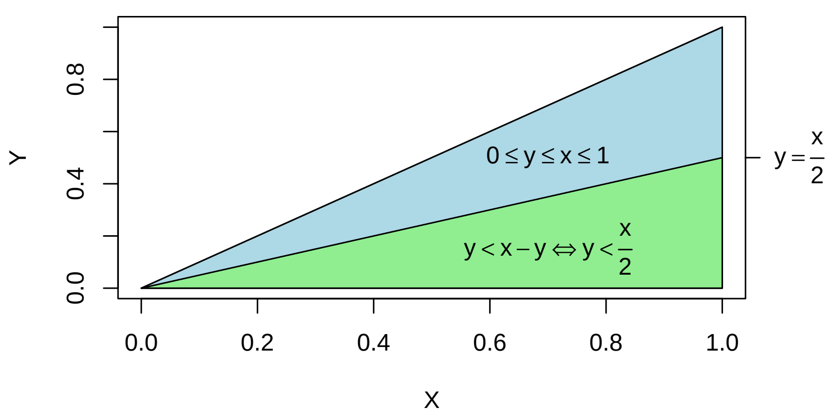
(c) (5 points) What is density function of the conditional distribution of $Y|X = x$?

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{8xy}{4x^3} = \frac{2y}{x^2}; \qquad 0 \le y \le x
\end{aligned}
$$

(d) (5 points) What is the probability that the first task takes less time to perform than the second?

If the first task takes less time than the second then

$$Y < X - Y \quad \Longleftrightarrow \quad 2Y < X \quad \Longleftrightarrow \quad Y < \frac{X}{2}$$



$$
\begin{aligned}
P[Y < 0.5X] &= \int_0^1 \int_0^{x/2} 8xy\,dy\,dx \\
&= \int_0^1 4x \left(\frac{x}{2}\right)^2 dx \\
&= \int_0^1 x^3 dx = \frac{1}{4}
\end{aligned}
$$

3. (10 points) Suppose $X \sim Exp(\lambda)$ and let $Y = \sqrt{X}$.

   (a) (5 points) What is the density function of the random variable Y?

   $$g^{-1}(y) = y^2 \qquad \frac{d}{dy} g^{-1}(y) = 2y$$

   $$
   \begin{aligned}
   f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
   &= 2y\lambda e^{-\lambda y^2}; \qquad y \geq 0
   \end{aligned}
   $$

   (b) (5 points) Suppose that the pth quantile of X is x_p. Show that y_p, the pth quantile of Y, satisfies $y_p = \sqrt{x_p}$.

   $$p = P[Y \leq y_p] = P[X \leq y_p^2] = P[X \leq x_p]$$

   This implies that $y_p^2 = x_p$ or $y_p = \sqrt{x_p}$.

   Aside: Note that the X having an exponential distribution was not used. In fact the result holds for any continuous random variable X that only takes non-negative values (e.g Betas, Gammas, lognormals, X from problem 2, etc.)

4. (15 points) A fast food restaurant chain is running a game where customers are given a letter (I, N, or W) with each purchase. When a customer has collected all 3 letters, spelling WIN, they receive a prize. Assume that each letter is equally likely at each visit.

   (a) (5 points) What is the probability that a customer will win with only 3 visits?

   Let Z be the total number of purchases. Then

   $$
   \begin{aligned}
   P[Z = 3] &= P[\text{New letter on 2nd visit}]P[\text{Final letter on 3rd visit}|\text{New letter on 2nd visit}] \\
   &= \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}
   \end{aligned}
   $$

   Or, note that there are $27 (= 3^3)$ different sets of 3 letters you can get with 3 purchase, if the order received in taken into account. Of those, 6 consist of 3 different letters, giving the probability $\frac{6}{27} = \frac{2}{9}$.

(b) (5 points) What is the probability that it takes a customer exactly 4 visits to win. (Hint: Let X be the number of purchases to receive the "second" letter (different than the one received on the first purchase), Y be the number of purchases needed to receive the final letter after receiving the "second" letter, and Z be the total number of purchases.)

$Z = 1 + X + Y$, so for $Z = 4$, $X = 1, Y = 2$ or $X = 2, Y = 1$.

$$\begin{aligned} P[Z=4] &= P[X=1, Y=2] + P[X=2, Y=1] \\ &= P[NRN] + P[RNN] \qquad (N = \text{New letter}, R = \text{Repeat letter}) \\ &= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} \\ &= \frac{4}{27} + \frac{2}{27} = \frac{6}{27} = \frac{2}{9} \end{aligned}$$

(c) (5 points) What is the expected number of visits required to collect all three letters? (Hint: see previous hint)

$X \sim Geom(p = 2/3)$ and $Y \sim Geom(p = 1/3)$.

$$E[Z] = 1 + E[X] + E[Y] = 1 + \frac{1}{2/3} + \frac{1}{1/3} = 1 + 1.5 + 3 = 5.5$$