**Statistics 110 – Quiz 2 Solutions** **Wednesday, August 2, 2006**

1. (10 points) Let $X \sim Pois(10)$ and $Y \sim Pois(5)$ be independent random variables. You may assume that the moment generating function for a $Pois(\lambda)$ random variable is $M(t) = \exp(\lambda(e^t - 1))$.

   (a) (5 points) By the use of moment generating functions, show that $Z = X+Y \sim Pois(15)$

   $$\begin{aligned} M_Z(t) &= M_X(t)M_Y(t) \\ &= \exp(5(e^t-1))\exp(10(e^t-1)) \\ &= \exp(15(e^t-1)) \end{aligned}$$

   This is the moment generating function for a $Pois(15)$ random variable.

   (b) (5 points) Verify that $E[Z] = 15$ by the use of the moment generating function.

   $$\begin{aligned} E[Z] &= M_Z'(0) \\ &= 15e^t \exp(15(e^t-1))\Big|_{t=0} \\ &= 15 \end{aligned}$$

2. (10 points) Let $Y$ be a random variable with $P[Y = 1] = P[Y = 2] = 0.5$. Given $Y$, the conditional distribution of $X$ is exponential with mean $Y$.

   (a) (5 points) Find $E[X]$.

   $$\begin{aligned} E[X] &= E[E[X|Y]] \\ &= E[Y] \\ &= \frac{1+2}{2} = 1.5 \end{aligned}$$

   (b) (5 points) Find $\text{Var}(X)$. (Hint: What is $\text{Var}(Y)$ and what is the relationship between $\text{Var}(Y)$ and $\text{Var}(E[X|Y])$?)

   $$\text{Var}(Y) = \frac{1}{2}(1-1.5)^2 + \frac{1}{2}(2-1.5)^2 = 2 \times \frac{1}{8} = \frac{1}{4}$$

   Note that this is the same as $\text{Var}(E[X|Y])$. Also note that the variance of an exponential is the mean squared. Then

   $$\begin{aligned} \text{Var}(X) &= E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) \\ &= E[Y^2] + \text{Var}(Y) \\ &= \text{Var}(Y) + (E[Y])^2 + \text{Var}(Y) \\ &= 2\text{Var}(Y) + (E[Y])^2 \\ &= 2\frac{1}{4} + \left(\frac{3}{2}\right)^2 \\ &= \frac{2+9}{4} = 2.75 \end{aligned}$$

3. (15 points) Let $X_1, X_2, X_3$, and $X_4$ be an independent random sample from a N(2,4) distribution and $Y_1, Y_2, \ldots, Y_9$, be an independent sample from a $N(1,9)$ distribution.

(a) (5 points) What is $E[\bar{X} - \bar{Y}]$?

Note that $\bar{X} \sim N(2,1)$ and $\bar{Y} \sim N(1,1)$. Thus

$$E[\bar{X} - \bar{Y}] = E[\bar{X}] - E[\bar{Y}] = 2 - 1 = 1$$

(b) (5 points) What is $\mathrm{Var}(\bar{X} - \bar{Y})$?

$$\mathrm{Var}(\bar{X} - \bar{Y}) = \mathrm{Var}(\bar{X}) + \mathrm{Var}(\bar{Y}) = 1 + 1 = 2$$

(c) (5 points) What is $P[\bar{X} \leq \bar{Y}]$? (Hint: How does $\bar{X} \leq \bar{Y}$ relate to $\bar{X} - \bar{Y}$?)

$$\begin{aligned} P[\bar{X} \leq \bar{Y}] &= P[\bar{X} - \bar{Y} \leq 0] \\ &= P\left[\frac{\bar{X} - \bar{Y} - 1}{\sqrt{2}} \leq \frac{0 - 1}{\sqrt{2}}\right] \\ &= P[Z \leq -0.707] = 0.2389 \end{aligned}$$