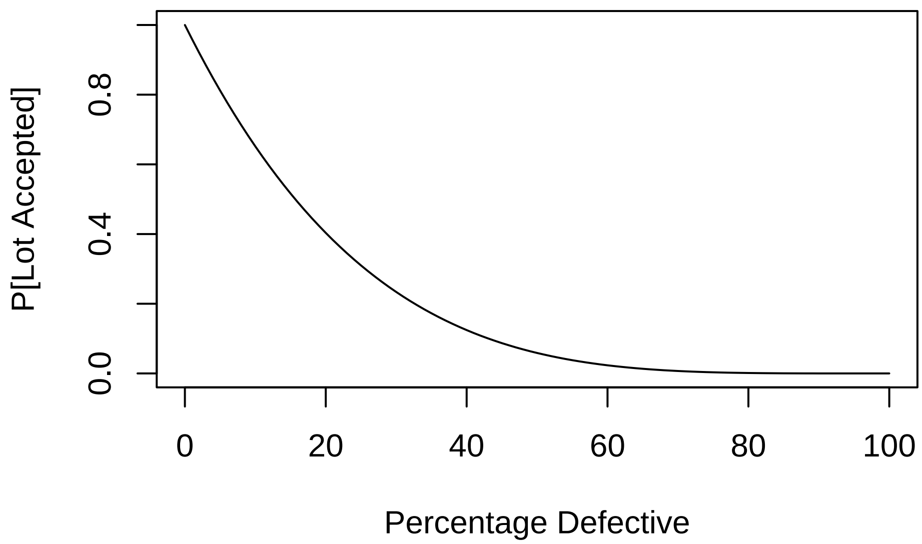
Statistics 110 – Assignment 1 Solutions **Summer, 2006**

1. Rice 1.17

   Let k be the number of defective items in the batch of 100 and define $p = P[\text{item selected is defective}] = \frac{k}{100}$. Then

$$\begin{aligned} P[\text{the lot is accepted}] &= P[\text{all the 4 selected items are not defective}] \\ &= \frac{\binom{100-k}{4}}{\binom{100}{4}} \\ &= \frac{100-k}{100} \times \frac{99-k}{99} \times \frac{98-k}{98} \times \frac{97-k}{97} \end{aligned}$$



2. Rice 1.20

   There are only 49 ways to place 4 aces which are all next to each other without considering their order. And there are $\binom{52}{4}$ ways to place 4 aces out of a deck of 52 cards. So the answer is

$$\frac{49}{\binom{52}{4}} \approx 1.81 \times 10^{-4}.$$

3. Rice 1.22

   There are $4 \times 3 = 16$ face cards (if aces included as face cards), so

$$\begin{aligned} P[\text{No face card is turned up}] &= \frac{\binom{52-16}{n}}{\binom{52}{n}} \\ &= \frac{36}{52} \times \frac{35}{51} \times \cdots \times \frac{36-n+1}{52-n+1} \end{aligned}$$

The probability of at least one face card is turned up among the n cards is

$$p_n = 1 - \frac{36}{52} \times \frac{35}{51} \times \cdots \times \frac{36-n+1}{52-n+1}$$

n	p_n
1	0.3077
2	0.5249
3	0.6770
4	0.7824

So n needs to be 2 for this probability to be about 0.5.

If aces are not considered as face cards, then

$$p_n = 1 - \frac{\binom{40}{n}}{\binom{52}{n}}$$

and

n	p_n
1	0.2308
2	0.4118
3	0.5529
4	0.6624

So n needs to be 3 for this probability to be about 0.5 in this case.

4. (a)

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\ &\leq P[A] + P[B] \end{aligned}$$

since $P[A \cap B] \geq 0$. Another approach is to note the following three facts

i. $A \cup B = (A \cap B^c) \cup B$
ii. $(A \cap B^c) \subset A \Rightarrow P[A \cap B^c] \leq P[A]$
iii. $A \cap B^c$ and B are disjoint sets

Then

$$P[A \cup B] = P[A \cap B^c] + P[B] \leq P[A] + P[B]$$

(b) In (a), we showed the inequality holds when $n = 2$. Suppose for $n \le k$, the equation also holds. Then for $n = k + 1$,

$$\begin{aligned} P[A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}] &= P[(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}] \\ &\le P[A_1 \cup A_2 \cup \cdots \cup A_k] + P[A_{k+1}] \\ &\qquad \text{(since it holds for 2 sets)} \\ &\le P[A_1] + P[A_2] + \cdots + P[A_k] + P[A_{k+1}] \\ &\qquad \text{(by the induction hypothesis)} \end{aligned}$$

So the inequality must hold for $n = k + 1$.

5. Rice 1.50, 1.51

A_i = the face value of dice i, and $i = 1, 2$.

$$\begin{aligned} P[\{A_1 = 3\} \cup \{A_2 = 3\} | A_1 + A_2 = 6] &= \frac{P[(\{A_1 = 3\} \cup \{A_2 = 3\}) \cap \{A_1 + A_2 = 6\}]}{P[A_1 + A_2 = 6]} \\ &= \frac{P[A_1 = 3, A_2 = 3]}{\sum_{i=1}^{5} P[\{A_1 = i\} \cap \{A_2 = 6 - i\}]} \\ &= \frac{\frac{1}{6} \times \frac{1}{6}}{5 \times \frac{1}{6} \times \frac{1}{6}} = \frac{1}{5} \end{aligned}$$

Note that if $A_1 + A_2 < 6$, $\{A_1 = 3\} \cap \{A_2 = 3\} = \emptyset$, so

$$\begin{aligned} P[\{A_1 = 3\} \cup \{A_2 = 3\} | A_1 + A_2 < 6] &= P[A_1 = 3 | A_1 + A_2 < 6] + P[A_2 = 3 | A_1 + A_2 < 6] \\ &= 2P[A_1 = 3 | A_1 + A_2 < 6] \\ &= 2\frac{P[\{A_1 = 3\} \cap \{A_1 + A_2 < 6\}]}{P[A_1 + A_2 < 6]} \\ &= 2\frac{P[A_1 = 3, A_2 = 2] + P[A_1 = 3, A_2 = 1]}{\sum_{i=1}^{4} \sum_{j=1}^{5-i} P[A_1 = i] P[A_2 = j]} \\ &= 2\frac{2 \times \frac{1}{36}}{\frac{1}{36} \times (1 + 2 + 3 + 4)} = \frac{2}{5} \end{aligned}$$

6. Rice 1.54

Define the events R_0 = rain today, R_i = rain i days from now.

(a)

$$P[R_1] = P[R_1 | R_0] P[R_0] + P[R_1 | R_0^c] P[R_0^c] = \alpha p + (1 - \beta)(1 - p)$$

(b)

$$\begin{aligned}
P[R_2] &= P[R_2|R_1]P[R_1] + P[R_2|R_1^c]P[R_1^c] \\
&= P[R_2|R_1]P[R_1] + P[R_2|R_1^c](1 - P[R_1]) \\
&= \alpha P[R_1] + (1-\beta)(1 - P[R_1]) \\
&= (\alpha + \beta - 1)P[R_1] + 1 - \beta \\
&= (\alpha + \beta - 1)(\alpha p + (1-\beta)(1-p)) + (1-\beta) \\
&= (\alpha + \beta - 1)^2 p + (\alpha + \beta - 1)(1-\beta) + (1-\beta)
\end{aligned}$$

(c)

$$\begin{aligned}
P[R_n] &= P[R_n|R_{n-1}]P[R_{n-1}] + P[R_n|R_{n-1}^c]P[R_{n-1}^c] \\
&= \alpha P[R_{n-1}] + (1-\beta)(1 - P[R_{n-1}]) \\
&= (\alpha + \beta - 1)P[R_{n-1}] + (1-\beta) \\
&= (\alpha + \beta - 1)^2 P[R_{n-2}] + (\alpha + \beta - 1)(1-\beta) + (1-\beta) \\
&= (\alpha + \beta - 1)^n P[R_0] + (\alpha + \beta - 1)^{n-1}(1-\beta) + (\alpha + \beta - 1)^{n-2}(1-\beta) \\
&\qquad + \cdots + (1-\beta) \\
&= (\alpha + \beta - 1)^n p + \frac{1 - (\alpha + \beta - 1)^n}{1 - (\alpha + \beta - 1)}(1-\beta)
\end{aligned}$$

So $P[R_n] \to \frac{1-\beta}{(1-\beta)+(1-\alpha)} = p_\infty$.
Another way of seeing this limit, is that if it exists, it must satisfy

$$p_\infty = \alpha p_\infty + (1-\beta)(1-p_\infty)$$

Solving for p_∞ gives the above limit.

7. Rice 1.56

Define the events A_1 = the oldest is a girl, A_2 = the youngest is a girl. Assume that $P[A_1] = P[A_2] = 0.5$ (not quite true but close enough) and that A_1 and A_2 are independent.

$$P[A_1 \cap A_2 | A_1] = \frac{P[A_1 \cap A_2]}{P[A_1]} = \frac{P[A_1]P[A_2]}{P[A_1]} = \frac{1}{2}$$

$$P[A_1 \cap A_2 | A_1 \cup A_2] = \frac{P[A_1 \cap A_2]}{P[A_1 \cup A_2]}$$

Since $P[A_1 \cup A_2] = P[A_1 \cap A_2^c] + P[A_1^c \cap A_2] + P[A_1 \cap A_2] = \frac{3}{4}$, then above equation gives $\frac{1}{3}$.

8. Rice 1.60

   Let A_i = the product comes form ith shift, $i = 1, 2, 3$. Since all the shifts have the same productivity, $P[A_i] = \frac{1}{3}$, $i = 1, 2, 3$. Let D =the product is defective. Then

$$P[D] = \sum_{i=1}^{3} P[D|A_i]P[A_i] = \frac{1\% + 2\% + 5\%}{3} = 2.67\%.$$

$$P[A_3|D] = \frac{P[A_3 \cap D]}{P[D]} = \frac{\frac{5\%}{3}}{\frac{1\%+2\%+5\%}{3}} = 0.625.$$

9. Rice 1.62

$$\begin{aligned} P[A] &= P[A|E]P[E] + P[A|E^c]P[E^c] \\ &\geq P[B|E]P[E] + P[B|E^c]P[E^c] = P[B] \end{aligned}$$

10. Rice 1.76

   Let X_t be the number of people at time t. As given, $X_0 = 1$. If $X_t > 0$, then the number of people in the queue could decrease by 1 (service a person in line and nobody joins queue), increase by 1 (a person joins the queue but nobody is serviced), or could stay the same (nobody serviced & nobody joins or one serviced & one joins).

$$X_{t+1} = \begin{cases} X_t - 1 & \text{with probability } p(1-q) \\ X_t + 1 & \text{with probability } (1-p)q \\ X_t & \text{with probability } pq + (1-p)(1-q) \end{cases}$$

   If $X_t = 0$, then X_{t+1} could only be 0 or 1, yielding

$$X_{t+1} = \begin{cases} 0 & \text{with probability } 1-q \\ 1 & \text{with probability } q \end{cases}$$

   Then

$$\begin{aligned} P[X_1 = 0] &= p(1-q) \\ P[X_1 = 1] &= pq + (1-p)(1-q) \\ P[X_1 = 2] &= q(1-p) \end{aligned}$$

   Then

$$\begin{aligned} P[X_2 = 0] &= (1-q)P[X_1 = 0] + p(1-q)P[X_1 = 1] \\ &= p(1-q)^2 + p(1-q)(pq + (1-p)(1-q)) \end{aligned}$$

$$\begin{aligned}
P[X_2 = 1] &= qP[X_1 = 0] + (pq + (1-p)(1-q))P[X_1 = 1] + p(1-q)P[X_1 = 2] \\
&= pq(1-q) + (pq + (1-p)(1-q))^2 + pq(1-p)(1-q)
\end{aligned}$$

$$\begin{aligned}
P[X_2 = 2] &= (1-p)qP[X_1 = 1] + (pq + (1-p)(1-q))P[X_1 = 2] \\
&= (1-p)q(pq + (1-p)(1-q)) + (pq + (1-p)(1-q))q(1-p) \\
&= 2(1-p)q(pq + (1-p)(1-q))
\end{aligned}$$

$$\begin{aligned}
P[X_2 = 3] &= (1-p)qP[X_1 = 2] \\
&= (1-p)^2q^2
\end{aligned}$$

These probabilities could also be determined by constructing the tree structure for the two time points and collecting the leaves corresponding to $X_2 = 0, 1, 2,$ and 3.

11. Rice 1.78

In what follows let F represent the genotype of the father and M represent the genotype of the mother.

(a)

$$P[AA] = P[Aa \text{ parent transmits } A] = \frac{1}{2}$$

$$P[Aa] = P[Aa \text{ parent transmits } a] = \frac{1}{2}$$

(b)

$$\begin{aligned}
P[AA] &= P[AA|F = AA, M = AA]P[F = AA]P[M = AA] \\
&\quad + P[AA|F = AA, M = Aa]P[F = AA]P[M = Aa] \\
&\quad + P[AA|F = AA, M = aa]P[F = AA]P[M = aa] \\
&\quad + P[AA|F = Aa, M = AA]P[F = Aa]P[M = AA] \\
&\quad + P[AA|F = Aa, M = Aa]P[F = Aa]P[M = Aa] \\
&\quad + P[AA|F = Aa, M = aa]P[F = Aa]P[M = aa] \\
&\quad + P[AA|F = aa, M = AA]P[F = aa]P[M = AA] \\
&\quad + P[AA|F = aa, M = Aa]P[F = aa]P[M = Aa] \\
&\quad + P[AA|F = aa, M = aa]P[F = aa]P[M = aa] \\
&= 1 \times p^2 + 0.5 \times 2pq + 0 \times pr + 0.5 \times 2pq + 0.25 \times 4q^2 + 0 \times 2qr \\
&\quad + 0 \times pr + 0 \times 2qr + 0 \times r^2 \\
&= p^2 + 2pq + q^2 = (p+q)^2
\end{aligned}$$

$$
\begin{aligned}
P[Aa] = & P[Aa|F=AA,M=AA]P[F=AA]P[M=AA] \\
& + P[Aa|F=AA,M=Aa]P[F=AA]P[M=Aa] \\
& + P[Aa|F=AA,M=aa]P[F=AA]P[M=aa] \\
& + P[Aa|F=Aa,M=AA]P[F=Aa]P[M=AA] \\
& + P[Aa|F=Aa,M=Aa]P[F=Aa]P[M=Aa] \\
& + P[Aa|F=Aa,M=aa]P[F=Aa]P[M=aa] \\
& + P[Aa|F=aa,M=AA]P[F=aa]P[M=AA] \\
& + P[Aa|F=aa,M=Aa]P[F=aa]P[M=Aa] \\
& + P[Aa|F=aa,M=aa]P[F=aa]P[M=aa] \\
= & 0\times p^2 + 0.5\times 2pq + 1\times pr + 0.5\times 2pq + 0.5\times 4q^2 + 0.5\times 2qr \\
& + 1\times pr + 0.5\times 2qr + 0\times r^2 \\
= & 2(pq+pr+qr+q^2) = 2(p+q)(q+r)
\end{aligned}
$$

$$
\begin{aligned}
P[aa] = & P[aa|F=AA,M=AA]P[F=AA]P[M=AA] \\
& + P[aa|F=AA,M=Aa]P[F=AA]P[M=Aa] \\
& + P[aa|F=AA,M=aa]P[F=AA]P[M=aa] \\
& + P[aa|F=Aa,M=AA]P[F=Aa]P[M=AA] \\
& + P[aa|F=Aa,M=Aa]P[F=Aa]P[M=Aa] \\
& + P[aa|F=Aa,M=aa]P[F=Aa]P[M=aa] \\
& + P[aa|F=aa,M=AA]P[F=aa]P[M=AA] \\
& + P[aa|F=aa,M=Aa]P[F=aa]P[M=Aa] \\
& + P[aa|F=aa,M=aa]P[F=aa]P[M=aa] \\
= & 0\times p^2 + 0\times 2pq + 0\times pr + 0\times 2pq + 0.25\times 4q^2 + 0.5\times 2qr \\
& + 0\times pr + 0.5\times 2qr + 1\times r^2 \\
= & q^2 + 2qr + r^2 = (q+r)^2
\end{aligned}
$$

Let $x = p+q$ and $y = q+r$ and note that $x+y=1$. Then the probabilities defined in the previous part satisfy $P[AA] = x^2, P[Aa] = 2xy$, and $P[aa] = y^2$. Following the approach of the previous part using these probabilities for the parental genotypes, we get for the third generation

$$
\begin{aligned}
P[AA] = {} & P[AA|F = AA, M = AA]P[F = AA]P[M = AA] \\
& + P[AA|F = AA, M = Aa]P[F = AA]P[M = Aa] \\
& + P[AA|F = AA, M = aa]P[F = AA]P[M = aa] \\
& + P[AA|F = Aa, M = AA]P[F = Aa]P[M = AA] \\
& + P[AA|F = Aa, M = Aa]P[F = Aa]P[M = Aa] \\
& + P[AA|F = Aa, M = aa]P[F = Aa]P[M = aa] \\
& + P[AA|F = aa, M = AA]P[F = aa]P[M = AA] \\
& + P[AA|F = aa, M = Aa]P[F = aa]P[M = Aa] \\
& + P[AA|F = aa, M = aa]P[F = aa]P[M = aa] \\
= {} & 1 \times x^4 + 0.5 \times 2x^3y + 0 \times x^2y^2 + 0.5 \times 2x^3y + 0.25 \times 4x^2y^2 + 0 \times 2xy^3 \\
& + 0 \times x^2y^2 + 0 \times 2xy^3 + 0 \times y^4 \\
= {} & x^4 + 2x^3y + x^2y^2 = x^2(x^2 + 2xy + y^2) = x^2(x + y)^2 = x^2 = (p + q)^2
\end{aligned}
$$

$$
\begin{aligned}
P[Aa] = {} & P[Aa|F = AA, M = AA]P[F = AA]P[M = AA] \\
& + P[Aa|F = AA, M = Aa]P[F = AA]P[M = Aa] \\
& + P[Aa|F = AA, M = aa]P[F = AA]P[M = aa] \\
& + P[Aa|F = Aa, M = AA]P[F = Aa]P[M = AA] \\
& + P[Aa|F = Aa, M = Aa]P[F = Aa]P[M = Aa] \\
& + P[Aa|F = Aa, M = aa]P[F = Aa]P[M = aa] \\
& + P[Aa|F = aa, M = AA]P[F = aa]P[M = AA] \\
& + P[Aa|F = aa, M = Aa]P[F = aa]P[M = Aa] \\
& + P[Aa|F = aa, M = aa]P[F = aa]P[M = aa] \\
= {} & 0 \times x^4 + 0.5 \times 2x^3y + 1 \times x^2y^2 + 0.5 \times 2x^3y + 0.5 \times 4x^2y^2 + 0.5 \times 2xy^3 \\
& + 1 \times x^2y^2 + 0.5 \times 2xy^3 + 0 \times y^4 \\
= {} & 2(x^3y + 2x^2y^2 + xy^3) = 2xy(x^2 + 2xy + y^2) = 2xy(x + y)^2 = 2xy = 2(p + q)(q + r)
\end{aligned}
$$

$$
\begin{aligned}
P[aa] = & P[aa|F = AA, M = AA]P[F = AA]P[M = AA] \\
& + P[aa|F = AA, M = Aa]P[F = AA]P[M = Aa] \\
& + P[aa|F = AA, M = aa]P[F = AA]P[M = aa] \\
& + P[aa|F = Aa, M = AA]P[F = Aa]P[M = AA] \\
& + P[aa|F = Aa, M = Aa]P[F = Aa]P[M = Aa] \\
& + P[aa|F = Aa, M = aa]P[F = Aa]P[M = aa] \\
& + P[aa|F = aa, M = AA]P[F = aa]P[M = AA] \\
& + P[aa|F = aa, M = Aa]P[F = aa]P[M = Aa] \\
& + P[aa|F = aa, M = aa]P[F = aa]P[M = aa] \\
= & 0 \times x^4 + 0 \times 2x^3y + 0 \times x^2y^2 + 0 \times 2x^3y + 0.25 \times 4x^2y^2 + 0.5 \times 2xy^3 \\
& + 0 \times x^2y^2 + 0.5 \times 2xy^3 + 1 \times y^4 \\
= & x^2y^2 + 2xy^3 + y^4 = y^2(x^2 + 2xy + y^2) = y^2(x + y)^2 = y^2 = (q + r)^2
\end{aligned}
$$

(c) The approach for this is the same as for part (b), except that different probabilities are needed for the parental genotypes. What is needed is

$$P[F = i|\text{ survived to mate}] \text{ for } i = AA, Aa, aa$$

(similarly for the mother's genotype). For calculating the second generation genotype probabilities we need

$$P[\text{survived to mate}] = pu + 2qv + rw = c$$

This gives

$$P[F = i|\text{ survived to mate}] = \begin{cases} \frac{pu}{c} & i = AA \\ \frac{2qv}{c} & i = Aa \\ \frac{rw}{c} & i = aa \end{cases}$$

Then the second generation probabilities are given by

$$
\begin{aligned}
P[AA] = & \frac{1}{c^2} \left\{ 1 \times p^2u^2 + 0.5 \times 2pquv + 0 \times pruw + 0.5 \times 2pquv + 0.25 \times 4q^2v^2 \right. \\
& \left. + 0 \times 2qrvw + 0 \times pruw + 0 \times 2qrvw + 0 \times r^2w^2 \right\} \\
= & \frac{1}{c^2} \left\{ p^2u^2 + 2pquv + q^2v^2 \right\} = \left(\frac{pu + qv}{c} \right)^2
\end{aligned}
$$

$$
\begin{aligned}
P[Aa] = \frac{1}{c^2}\big\{&0 \times p^2u^2 + 0.5 \times 2pquv + 1 \times pruw + 0.5 \times 2pquv + 0.5 \times 4q^2v^2 \\
&+0.5 \times 2qrvw + 1 \times pruw + 0.5 \times 2qrvw + 0 \times r^2w^2\big\} \\
= \frac{1}{c^2}&\left\{2(pquv + pruw + qrvw + q^2v^2)\right\} = 2\left(\frac{pu+qv}{c}\right)\left(\frac{qv+rw}{c}\right)
\end{aligned}
$$

$$
\begin{aligned}
P[aa] = \frac{1}{c^2}\big\{&0 \times p^2u^2 + 0 \times 2pquv + 0 \times pruw + 0 \times 2pquv + 0.25 \times 4q^2v^2 \\
&+0.5 \times 2qrvw + 0 \times pruw + 0.5 \times 2qrvw + 1 \times r^2w^2\big\} \\
= \frac{1}{c^2}&\left\{q^2v^2 + 2qrvw + r^2w^2\right\} = \left(\frac{qv+rw}{c}\right)^2
\end{aligned}
$$

Similarly to getting the third generations in part (b), let $x = \frac{pu+qv}{c}$ and $y = \frac{qv+rw}{c}$ and again note that $x + y = 1$. Based on this, the needed parental genotype probabilities for the third generation calculations are

$$
P[\text{survived to mate}] = x^2u + 2xyv + y^2w = d
$$

$$
P[F = i|\text{ survived to mate}] = \begin{cases} \frac{x^2u}{d} & i = AA \\ \frac{2xyv}{d} & i = Aa \\ \frac{y^2w}{d} & i = aa \end{cases}
$$

Then the third generation probabilities are given by

$$
\begin{aligned}
P[AA] = \frac{1}{d^2}\big\{&1 \times x^4u^2 + 0.5 \times 2x^3yuv + 0 \times x^2y^2uw + 0.5 \times 2x^3yuv + 0.25 \times 4x^2y^2v^2 \\
&+0 \times 2xy^3vw + 0 \times x^2y^2uw + 0 \times 2xy^3vw + 0 \times y^4w^2\big\} \\
= \frac{1}{d^2}&\left\{x^4u^2 + 2x^3yuv + x^2y^2v^2\right\} = x^2\left(\frac{xu+yv}{d}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
P[Aa] = \frac{1}{d^2}\big\{&0 \times x^4u^2 + 0.5 \times 2x^3yuv + 1 \times x^2y^2uw + 0.5 \times 2x^3yuv + 0.5 \times 4x^2y^2v^2 \\
&+0.5 \times 2xy^3vw + 1 \times x^2y^2uw + 0.5 \times 2xy^3vw + 0 \times y^4w^2\big\} \\
= \frac{1}{d^2}&\left\{2(x^3yuv + x^2y^2uw + xy^3vw + x^2y^2v^2)\right\} = 2xy\left(\frac{xu+yv}{d}\right)\left(\frac{xv+yw}{d}\right)
\end{aligned}
$$

$$
\begin{aligned}
P[aa] &= \frac{1}{d^2}\left\{0 \times x^4u^2 + 0 \times 2x^3yuv + 0 \times x^2y^2uw + 0 \times 2x^3yuv + 0.25 \times 4x^2y^2v^2\right. \\
&\qquad \left. +0.5 \times 2xy^3vw + 0 \times x^2y^2uw + 0.5 \times 2xy^3vw + 1 \times y^4w^2\right\} \\
&= \frac{1}{d^2}\left\{x^2y^2v^2 + 2xy^3vw + y^4w^2\right\} = y^2\left(\frac{xv + yw}{d}\right)^2
\end{aligned}
$$

Note that these aren't the same as the second generation probabilities. There is a drift due differential survival rates for the different genotypes.

12.

$$
\begin{aligned}
P[A_2|A_1] &= \frac{P[A_1 \cap A_2]}{P[A_1]} \\
&= \frac{P[A_1 \cap A_2|\text{Female}]P[\text{Female}] + P[A_1 \cap A_1|\text{male}]P[\text{male}]}{P[A_1|\text{Female}]P[\text{Female}] + P[A_1|\text{male}]P[\text{male}]} \\
&= \frac{p_f^2(1-\alpha) + p_m^2\alpha}{p_f(1-\alpha) + p_m\alpha}
\end{aligned}
$$

The desire result, $P[A_2|A_1] > P[A_1]$ holds since

$$
P[A_1 \cap A_2] = p_f^2(1-\alpha) + p_m^2\alpha > (p_f(1-\alpha) + p_m\alpha)^2 = P[A_1]^2
$$

This inequality holds as

$$
\begin{aligned}
P[A_1 \cup A_2] - P[A_1]^2 &= p_f^2(1-\alpha) + p_m^2\alpha - (p_f(1-\alpha) + p_m\alpha)^2 \\
&= (p_f - p_m)^2\alpha(1-\alpha) > 0
\end{aligned}
$$