Statistics 110 – Assignment 2 Solutions **Summer, 2006**

1. Rice 2.13

   (a)

$$\begin{aligned} P[\text{the student passes}] &= P[X \geq 12] \\ &= \sum_{i=12}^{20} P[X = i] \\ &= \sum_{i=12}^{20} \binom{20}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{20-i} \\ &= 1 - P[X \leq 11] = 0.0130 \end{aligned}$$

   (b)

$$\begin{aligned} P[\text{the student passes}] &= P[X \geq 12] \\ &= \sum_{i=12}^{20} P[X = i] \\ &= \sum_{i=12}^{20} \binom{20}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{20-i} \\ &= 1 - P[X \leq 11] = 0.2517 \end{aligned}$$

2. Rice 2.14

   (a) Let X is the total number of attempts. Then

$$P[X = n] = \begin{cases} (1-P_1)^{\frac{n}{2}}(1-P_2)^{\frac{n}{2}-1}P_2 & \text{if } n \text{ is even, that is, player B wins} \\ (1-P_1)^{\frac{n-1}{2}}(1-P_2)^{\frac{n-1}{2}}P_1 & \text{if } n \text{ is odd, that is, player A wins} \end{cases}$$

   (b)

$$\begin{aligned} P[\text{A wins}] = P[X \text{ is odd}] &= \sum_{n \text{ is odd}} (1-P_1)^{\frac{n-1}{2}}(1-P_2)^{\frac{n-1}{2}}P_1 \\ &= P_1 \sum_{k=0}^{\infty} [(1-P_1)(1-P_2)]^k \quad \left(\text{let } k = \frac{n-1}{2}\right) \\ &= \frac{P_1}{1-(1-P_1)(1-P_2)} = \frac{P_1}{P_1 + P_2 - P_1 P_2} \end{aligned}$$

3. Rice 2.20

$$P[X \le k] = \sum_{i=1}^{k} (1-p)^{i-1} p = p\frac{1-(1-p)^k}{p} = 1-(1-p)^k$$
$$= 1 - \left(\frac{1}{2}\right)^k$$

Then $1 - \left(\frac{1}{2}\right)^k = 0.99$, so $k = [\frac{\ln 100}{\ln 2}] + 1 = 7$ ($k = 6$ is also ok).

4. Rice 2.46

$$P[T < 1] = \int_0^1 \lambda e^{-\lambda x} dx = 1 - e^{-\lambda} = 0.05$$

So $\lambda = \log \frac{100}{95} = 0.05129329$

5. Rice 2.53

(a)

$$P[X > 10] = P\left[\frac{X-5}{10} > \frac{10-5}{10}\right]$$
$$= P\left[Z > 0.5\right]$$
$$= 1 - P[Z \le 0.5]$$
$$= 1 - \Phi(0.5) = 0.3085$$

(b)

$$P[-20 < X < 15] = P\left[\frac{-20-5}{10} < \frac{X-5}{10} < \frac{15-5}{10}\right]$$
$$= P[-2.5 < Z < 1]$$
$$= \Phi(1) - \Phi(-2.5) = 0.8351$$

(c) Since $P[X > x] = 1 - \Phi(\frac{x-5}{10}) = 0.05$, then $\Phi(\frac{x-5}{10}) = 0.95$. From the table 2 $\frac{x-5}{10} = 1.645$ (the cell in table closest to 0.95), implying

$$x = 10 \times 1.645 + 5 = 21.45$$

6.

$$\begin{aligned} P[\mu - c \le X \le \mu + c] &= P[\frac{-c}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{c}{\sigma}] \\ &= P[\frac{-c}{\sigma} \le Z \le \frac{c}{\sigma}] \\ &= \Phi\left(\frac{c}{\sigma}\right) - \Phi\left(\frac{-c}{\sigma}\right) \\ &= 2\Phi\left(\frac{c}{\sigma}\right) - 1 = 0.2 \end{aligned}$$

So $\Phi\left(\frac{c}{\sigma}\right) = 0.6 \Rightarrow \frac{c}{\sigma} = 0.25$ (the cell in the normal table closest to 0.6) which gives $c = 0.25\sigma$

7. (a)

$$\int_1^\infty \frac{c}{x^{\alpha+1}} dx = -\left.\frac{c}{\alpha x^\alpha}\right|_1^\infty = \frac{c}{\alpha} = 1$$

so $c = \alpha$

(b) In the integration in (a), x^α needs to go to ∞ when $x \to \infty$, so $\alpha > 0$.

(c) Let M be the median and $Q1$ and $Q3$ be the lower and upper quartiles. Then

$$\int_1^M \frac{\alpha}{x^{\alpha+1}} dx = 1 - \frac{1}{M^\alpha} = \frac{1}{2}$$

Then $M = 2^{1/\alpha}$.
For the upper quartile

$$\int_1^{Q3} \frac{\alpha}{x^{\alpha+1}} dx = 1 - \frac{1}{(Q3)^\alpha} = \frac{3}{4} \Rightarrow \frac{1}{(Q3)^\alpha} = \frac{1}{4}$$

Then $Q3 = 4^{1/\alpha}$ and similarly $Q1 = \left(\frac{4}{3}\right)^{1/\alpha}$.

8. Rice 4.4

$f(x) = F'(x) = \alpha x^{-\alpha-1}$, $x \ge 1$, which is the same density function as in the previous problem.

(a)

$$\begin{aligned} E[X] &= \int_1^\infty x \alpha x^{-\alpha-1} dx \\ &= \int_1^\infty \alpha x^{-\alpha} dx \\ &= \alpha \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_1^\infty \\ &= \frac{\alpha}{\alpha-1} \qquad \text{if } \alpha > 1 \end{aligned}$$

(b)

$$E[X^2] = \int_1^\infty x^2 \alpha x^{-\alpha-1} dx = \frac{\alpha}{\alpha - 2} \qquad \text{if } \alpha > 2$$

So if $\alpha > 2$,

$$\begin{aligned}\text{Var}(X) &= E[X^2] - E[X]^2 \\ &= \frac{\alpha}{\alpha-2} - \left(\frac{\alpha}{\alpha-1}\right)^2 \\ &= \frac{\alpha}{(\alpha-2)(\alpha-1)^2}\end{aligned}$$

9. Rice 4.6

(a)

$$E[X] = \int_0^1 x 2x dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

(b) The CDF of X satisfies

$$F_X(x) = \int_0^x 2u du = u^2\big|_0^x = x^2$$

$$F_Y(y) = P[Y \le y] = P[X^2 \le y] = P[0 \le X \le \sqrt{y}] = (\sqrt{y})^2 = y$$

then $f_Y(y) = F'_Y(y) = 1$, for $0 \le y \le 1$ ($Y \sim U(0,1)$). Then

$$E[Y] = \int_0^1 y \cdot 1 dy = \frac{1}{2}$$

(c)

$$\begin{aligned}E[X^2] &= \int_0^1 x^2 \cdot 2x dx \\ &= \int_0^1 2x^3 dx \\ &= \frac{1}{2}\end{aligned}$$

which is the same as $E(Y)$.

(d)

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

10. Rice 4.10

”Each item is equally likely to be the one requested” is equivalent to that the one requested tends to be located at each of the n places with equal probability. Thus X, the number of items are searched through in the list until find the one requested is uniformly distributed on $1, 2, \cdots, n$. Then

$$P[X = i] = \frac{1}{n}, \qquad i = 1, 2, \cdots, n$$

So

$$E[X] = \sum_{i=1}^{n} \frac{i}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$$

which is the midpoint of the possible locations.

11. (a)

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + p = \frac{7}{12} + p = 1$$

so $p = \frac{5}{12}$.

(b)

$$E[X] = 1\frac{1}{3} + 2\frac{1}{6} + 3\frac{1}{12} + 4\frac{5}{12} = \frac{31}{12}$$

(c)

$$E[X^2] = 1^2\frac{1}{3} + 2^2\frac{1}{6} + 3^2\frac{1}{12} + 4^2\frac{5}{12} = \frac{101}{12}$$

So

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{101}{12} - \left(\frac{31}{12}\right)^2 = \frac{251}{144}$$

Suggested Problems

1. Rice 2.11

If k_{max} is the choice of k that maximizes $P[X = k]$, then

$$\begin{aligned}\frac{P[X = k_{max}]}{P[X = k_{max} - 1]} &= \frac{\binom{n}{k_{max}} p^{k_{max}} (1-p)^{n-k_{max}}}{\binom{n}{k_{max}-1} p^{k_{max}-1} (1-p)^{n-k_{max}+1}} \\ &= \frac{n - k_{max} + 1}{k} \frac{p}{1-p} \\ &\geq 1\end{aligned}$$

and

$$\begin{aligned}\frac{P[X = k_{max} + 1]}{P[X = k_{max}]} &= \frac{\binom{n}{k_{max}+1} p^{k_{max}+1} (1-p)^{n-k_{max}-1}}{\binom{n}{k_{max}} p^{k_{max}} (1-p)^{n-k_{max}}} \\ &= \frac{n - k_{max} + 1}{k} \frac{p}{1-p} \\ &\leq 1\end{aligned}$$

These two inequalities yield

$$\frac{n - k_{max}}{k_{max} + 1} \leq \frac{p}{1-p} \leq \frac{n - k_{max} + 1}{k_{max}}$$

Solving for k_{max} in the above gives

$$k_l = \frac{n - \frac{p}{1-p}}{\frac{p}{1-p} + 1} \leq k_{max} \leq \frac{n+1}{\frac{p}{1-p} + 1} = k_u$$

Note that it can be shown that the upper limit is 1 more than the lower limit so k_{max} will be unique integer in the interval unless the endpoints happen to be integers. Then you need to see which endpoint gives the maximum, though it is possible that both endpoints will give the same probability (i.e. $P[X = k_l] = P[X = k_u]$)

For example, if $p = 0.75$ and $n = 20$, $14.75 \leq k_{max} \leq 15.75$ so $k_{max} = 15$.

2. Rice 2.21

   For $X \sim Geo(p)$

$$F_X(k) = P[X \leq k] = \sum_{r=1}^{k} p(1-p)^{r-1} = p\frac{1-(1-p)^k}{1-(1-p)} = 1 - (1-p)^k$$

$$\begin{aligned} P[X > n+k-1 | X > n-1] &= \frac{P[\{X > n+k-1\} \cap \{X > n-1\}]}{P[X > n-1]} \\ &= \frac{P[X > n+k-1]}{P[X > n-1]} \\ &= \frac{1 - P[X \leq n+k-1]}{1 - P[X \leq n-1]} \\ &= \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}} \\ &= (1-p)^k \\ &= 1 - P[X \leq k] = P[X > k] \end{aligned}$$

3. Rice 2.30

   (a) For 1 year, $\lambda = 12 \times 0.33 = 4$. Then

$$P[X = k] = \frac{e^{-4}4^k}{k!}$$

   To find the mode, we can mimic the solution to Rice 2.11. This yields for general $Pois(\lambda)$

$$\frac{\lambda}{k_{max} + 1} \leq 1 \leq \frac{\lambda}{k_{max} + 1}$$

Rearranging gives

$$\lambda - 1 \le k_{max} \le \lambda$$

In the case when λ is an integer (as it is here)

$$P[X = \lambda - 1] = P[X = \lambda]$$

so the most likely number of suicides is 3 or 4.
Note: If you take $\lambda = 3.96$ (don't treat 0.33 as shorthand for $\frac{1}{3}$), the most likely number of suicides is 3.

(b) For one week, $\lambda = \frac{0.33}{4} = 0.0825$

$$P[Y = 2] = \frac{0.0825^2 e^{-0.0825}}{2!} = 0.0031$$

4. Rice 2.31

(a) $\lambda = \frac{2}{6} = \frac{1}{3}$

$$\begin{aligned} P[\text{phone rings}] &= P[X > 0] \\ &= 1 - P[X = 0] \\ &= 1 - \frac{e^{-1/3}\left(\frac{1}{3}\right)^0}{0!} \\ &= 1 - e^{-1/3} = 0.283 \end{aligned}$$

(b) Let t be the length of her shower in hours. Then the associated $\lambda = 2t$. Need to select t such that

$$P[\text{phone rings}] = P[\text{phone doesn't ring}] = 0.5$$

So we need to solve

$$\begin{aligned} P[X = 0] &= e^{-2t} = 0.5 \\ &\Rightarrow -2t = \log 0.5 \\ &\Rightarrow t = -0.5 \log 0.5 = 0.347 \text{ hours} \end{aligned}$$

which corresponds to 20.79 minutes.

5. Rice 2.36 If U is in the range [0,1], then nU takes values in the range [0,n]. Then X takes values

$$X = \begin{cases} 0 & U \in [0, \frac{1}{n}) \\ 1 & U \in [\frac{1}{n}, \frac{2}{n}) \\ & \dots \\ k & U \in [\frac{k}{n}, \frac{k+1}{n}) \\ & \dots \\ n-1 & U \in [\frac{n-1}{n}, 1) \\ n & U = 1 \end{cases}$$

So for k = 0, 1, . . . , n - 1

$$P[X = k] = P\left[\frac{k}{n} \leq U < \frac{k+1}{n}\right] = \frac{1}{n}$$

and $[X = n] = P[U = 1] = 1$

6. Rice 2.45

   (a)

$$P[X < 10] = 1 - e^{-0.1\times 10} = 1 - e^{-1} = 0.6321$$

   (b)

$$\begin{aligned} P[5 < X < 15] &= P[X < 15] - P[X < 5] \\ &= (1 - e^{-0.1\times 15}) - (1 - e^{-0.1\times 5}) \\ &= e^{-0.5} - e^{-1.5} = 0.383 \end{aligned}$$

7. Rice 4.9 Suppose that n items are to be stocked and let Y be the random variable representing the number sold. Then if k items are requested to be sold, the net income Z is

$$Z = \begin{cases} sk - cn & \text{if } k \leq n \qquad \text{(can supply all requests from the } n \text{ items in stock)} \\ sn - cn & \text{if } k > n \qquad \text{(can sell at most the } n \text{ items in stock)} \end{cases}$$

   Then the expected net income given n items stocked satisfies

$$E_x[Z] = \sum_{k=0}^{n}(sk - cn)p(k) + (sn - cn)P[Y > n] \quad = \left(\sum_{k=0}^{n} skp(k) + snP[Y > n]\right) - cn$$

   For the maximum net incomes, n_{opt} satisfies

$$E_{n_{opt}}[Z] - E_{n_{opt}-1}[Z] \geq 0 \quad \text{and} \quad E_{n_{opt}+1}[Z] - E_{n_{opt}1}[Z] \leq 0$$

   These reduce to

$$sP[Y \geq n_{opt}] \geq c \quad \text{and} \quad sP[Y \geq n_{opt} + 1] \leq c$$

8. Rice 4.21 $X \sim U(0, 1)$. Then

$$E[X^2] = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$