Statistics 110 – Assignment 4 Solutions **Summer, 2006**

1. Rice 3.46

$$f_{X,Y}(x,y) = e^{-x-y}; x \geq 0, y \geq 0.$$

Let $r = \sqrt{x^2+y^2}$ and $\theta = \arctan\frac{y}{x}$ for $0 \leq \theta < 2\pi$ and $r \geq 0$, then $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$J = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

So

$$|J| = \frac{x}{\sqrt{x^2+y^2}}\frac{x}{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}\frac{y}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Giving

$$f_{R,\theta}(r,\theta) = re^{-(r\cos\theta + r\sin\theta)}; 0 \leq \theta < 2\pi, r \geq 0.$$

As $f(r,\theta)$ can not be expressed in a product of a function of r and another function of θ, R and θ are not independent.

2. Rice 3.54

Let $U = X + Y$ and $V = \frac{X}{Y}$, then $Y = \frac{U}{V+1}$ and $X = \frac{UV}{V+1}$.

$$J = \begin{bmatrix} 1 & 1 \\ \frac{1}{y} & \frac{-x}{y^2} \end{bmatrix}$$

Then

$$|J| = \frac{1}{y} + \frac{x}{y^2} = \frac{x+y}{y^2} = \frac{u}{u^2/(v+1)^2} = \frac{(v+1)^2}{u}$$

So

$$\begin{aligned} f_{U,V}(u,v) &= \lambda^2 e^{-\lambda uv/(v+1)} e^{-\lambda u/v+1} \frac{u}{(v+1)^2} \\ &= \left(\lambda^2 u e^{-\lambda u}\right)\left(\frac{1}{(v+1)^2}\right) \end{aligned}$$

for $u \geq 0, v \geq 0$. So U and V are independent.

3. Rice 3.55

   Let X_i be the life time for the ith component, then $X_i \sim Exp(\lambda_i)$. Because of series connection, the system will only work when every component is works, so the life time of the system $Y = \min(X_1, \ldots, X_n)$

$$\begin{aligned} P(Y > y) &= \prod_{i=1}^{n} P[X_i > y] = \prod_{i=1}^{n} (e^{-\lambda_i y}) \\ &= e^{(\Sigma_{i=1}^{n} \lambda_i) y}, \ y \geq 0 \end{aligned}$$

   So

$$f_Y(y) = (\Sigma_{i=1}^{n} \lambda_i) e^{(\Sigma_{i=1}^{n} \lambda_i) y}; y \geq 0$$

   implying the life time of the system is exponential with parameter $\Sigma_{i=1}^{n} \lambda_i$.

4. Rice 3.56

   Let Y_i be the lifetime for each parallel line. From the result in 3.55 above, we know that $Y_i \sim Exp(2\lambda)$, $i = 1, 2, 3$. Let Z is the lifetime for the system. Because of parallel connection, $Z = \max(Y_1, Y_2, Y_3)$ since the system will work as long as one line is working. So

$$P[Z \leq z] = \prod_{i=1}^{3} P[Y_i \leq z] = (1 - e^{-2\lambda z})^3; z \geq 0$$

$$f_Z(z) = 6\lambda e^{-2\lambda z}(1 - e^{-2\lambda z})^2; z \geq 0.$$

5. Rice 3.60

$$P[0.25 \leq X_i \leq 0.75] = \frac{1}{2}$$

   so

$$P[\text{All } 0.25 \leq X_i \leq 0.75] = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

6. Rice 4.17

   Since

$$\begin{aligned} f_{X_{(k)}}(x) &= \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1 - F(x)]^{n-k} \\ &= \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \end{aligned}$$

So

$$\begin{aligned}
E[X_{(k)}] &= \int_0^1 \frac{n!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} dx \\
&= \int_0^1 \frac{n!}{(k-1)!(n-k)!} \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \\
&= \frac{k}{n+1}
\end{aligned}$$

$$\begin{aligned}
E[(X_{(k)})^2] &= \int_0^1 \frac{n!}{(k-1)!(n-k)!} x^{k+1} (1-x)^{n-k} dx \\
&= \int_0^1 \frac{n!}{(k-1)!(n-k)!} \frac{\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+3)} \\
&= \frac{k(k+1)}{(n+1)(n+2)}
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(X_{(k)}) &= \frac{k(k+1)}{(n+1)(n+2)} - \left(\frac{k}{n+1}\right)^2 \\
&= \frac{k(n+1-k)}{(n+1)^2(n+2)}
\end{aligned}$$

Another approach is to notice that the density is that of a $Beta(k, n-k+1)$ distribution, which has the moments given above.

7. Rice 4.45

   (a)

$$E[Z] = E[\alpha X + (1-\alpha)Y] = \alpha E[X] + (1-\alpha)E[Y] = \mu$$

   (b)

$$\begin{aligned}
\text{Var}(Z) &= \text{Var}(\alpha X + (1-\alpha)Y) = \text{Var}(\alpha X) + \text{Var}((1-\alpha)Y) \\
&= \alpha^2 \text{Var}(X) + (1-\alpha)^2 \text{Var}(Y) \\
&= \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2
\end{aligned}$$

   This is minimized by $\alpha = \frac{\sigma_y^2}{\sigma_x^2 + \sigma_y^2}$ as

$$\frac{d}{d\alpha} \text{Var}(Z) = 2\alpha\sigma_x^2 - 2(1-\alpha)\sigma_y^2 = 2\alpha(\sigma_x^2 + \sigma_y^2) - 2\sigma_y^2$$

(c)

$$\text{Var}\left(\frac{X+Y}{2}\right) = \frac{\sigma_x^2 + \sigma_y^2}{4} \leq \sigma_x^2 \text{ if } \sigma_y^2 \leq 3\sigma_x^2.$$

Similarly $\text{Var}(\frac{X+Y}{2}) \leq \sigma_y^2$ if $\sigma_x \leq 3\sigma_y^2$.

So it is better to use $\frac{X+Y}{2}$ when $\frac{1}{3} \leq \frac{\sigma_y^2}{\sigma_x^2} \leq 3$.

8. Rice 4.48

$$\begin{aligned}\text{Cov}(U,V) &= \text{Cov}(Z+X, Z+Y) = \text{Cov}(Z,Z) + \text{Cov}(X,Z) + \text{Cov}(Z,Y) + \text{Cov}(X,Y) \\ &= \text{Var}(Z) = \sigma_Z^2\end{aligned}$$

Next,

$$\text{Var}(U) = \text{Var}(Z+X) = \text{Var}(Z) + \text{Var}(X) = \sigma_Z^2 + \sigma_X^2$$

Similarly, $\text{Var}(V) = \sigma_Z^2 + \sigma_Y^2$. So

$$\rho_{U,V} = \frac{\sigma_Z^2}{\sqrt{(\sigma_X^2 + \sigma_Z^2)(\sigma_Z^2 + \sigma_Y^2)}}$$

9. Rice 4.49

$$\begin{aligned}E[T] &= \sum_{k=1}^{n} kE[X_k] = \sum_{k=1}^{n} k\mu = \mu\sum_{k=1}^{n} k \\ &= \mu\frac{n(n+1)}{2}\end{aligned}$$

$$\begin{aligned}\text{Var}(T) &= \sum_{k=1}^{n} k^2\text{Var}(X_k) = \sum_{k=1}^{n} k^2\sigma^2 = \sigma^2\sum_{k=1}^{n} k^2 \\ &= \sigma^2\frac{n(n+1)(2n+1)}{6}\end{aligned}$$

10. Rice 4.50

$$\text{Var}(S) = \sum_{k=1}^{n} \text{Var}(X_k) = n\sigma^2$$

$$
\begin{aligned}
\text{Cov}(S,T) &= \sum_{j=1}^{n}\sum_{k=1}^{n}\text{Cov}(X_j, kX_k) \\
&= \sum_{k=1}^{n} k\text{Var}(X_k) \\
&= \sigma^2 \frac{n(n+1)}{2}
\end{aligned}
$$

$$
\begin{aligned}
\rho_{S,T} &= \frac{\text{Cov}(S,T)}{\sqrt{\text{Var}(S)\text{Var}(T)}} \\
&= \frac{\sigma^2 \frac{n(n+1)}{2}}{\sqrt{\sigma^2 n \times \sigma^2 \frac{n(n+1)(2n+1)}{6}}} \\
&= \sqrt{\frac{3(n+1)}{2(2n+1)}}
\end{aligned}
$$

11.

$$
\begin{aligned}
E[XY] &= \int_0^\infty \left[\int_0^x xy\frac{2e^{-2x}}{x}dy\right]dx \\
&= \int_0^\infty \left[y^2 e^{-2x}\Big|_0^x\right]dx \\
&= \int_0^\infty x^2 e^{-2x} dx \\
&= \int_0^\infty \frac{(2x)^2 e^{-2x}}{8} d(2x) \\
&= \frac{1}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
E[X] &= \int_0^\infty \left[\int_0^x 2e^{-2x} dy\right]dx \\
&= \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_0^\infty \left[\int_0^x \frac{2ye^{-2x}}{x}dy\right]dx \\
&= \int_0^\infty xe^{-2x} dx = \frac{1}{4}
\end{aligned}
$$

Since $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$, then

$$\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

Suggested Problems

1. Rice 3.48

   If (X_1, X_2) are bivariate normal, they have a density of the form

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_1}\right]\right)$$

$$\begin{array}{l} y_1 = g_1(x_1,x_2) = a_1x_1 + b_1 \\ y_2 = g_2(x_1,x_2) = a_2x_2 + b_2 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = h_1(y_1,y_2) = \frac{y_1-b_1}{a_1} \\ x_2 = h_2(y_1,y_2) = \frac{y_2-b_2}{a_2} \end{array}$$

$$J = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \qquad |J| = a_1a_2$$

Then

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}\left(\frac{y_1-b_1}{a_1}, \frac{y_2-b_2}{a_2}\right)\frac{1}{J} \\ &= \frac{1}{2\pi a_1\sigma_1 a_2\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(\frac{y_1-b_1}{a_1}-\mu_1)^2}{\sigma_1^2} + \frac{(\frac{y_2-b_2}{a_2}-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(\frac{y_1-b_1}{a_1}-\mu_1)(\frac{y_2-b_2}{a_2}-\mu_2)}{\sigma_1\sigma_2}\right]\right) \\ &= \frac{1}{2\pi a_1\sigma_1 a_2\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(y_1-a_1\mu_1-b_1)^2}{(a_1\sigma_1)^2} + \frac{(y_2-a_2\mu_2-b_2)^2}{(a_2\sigma_2)^2} - \frac{2\rho(x_1-a_1\mu_1-b_1)(x_2-a_2\mu_2-b_2)}{a_1\sigma_1a_2\sigma_2}\right]\right) \end{aligned}$$

which is a bivariate normal density with means $a_1\mu_1 + b_1$ and $a_2\mu_2 + b_2$, variances $(a_1\sigma_1)^2$ and $(a_2\sigma_2)^2$, and correlation ρ.

2. Rice 3.50

   In problem 3.49, part of the answer was to show that

$$
\begin{aligned}
E[Y_1] &= a_{11}E[X_1] + a_{12}E[X_2] + b_1 = b_1 = \mu_1 \\
E[Y_2] &= a_{21}E[X_1] + a_{22}E[X_2] + b_2 = b_2 = \mu_2 \\
\mathrm{Var}(Y_1) &= a_{11}^2\mathrm{Var}(X_1) + a_{12}\mathrm{Var}(X_2) = a_{11}^2 + a_{12}^2 = \sigma_1^2 \\
\mathrm{Var}(Y_2) &= a_{21}^2\mathrm{Var}(X_1) + a_{22}\mathrm{Var}(X_2) = a_{21}^2 + a_{22}^2 = \sigma_2^2 \\
\mathrm{Cov}(Y_1, Y_2) &= a_{11}a_{21}\mathrm{Var}(X_1) + a_{12}a_{22}\mathrm{Var}(X_2) = a_{11}a_{21} + a_{12}a_{22} = \rho\sigma_1\sigma_2 \\
\mathrm{Corr}(Y_1, Y_2) &= \frac{a_{11}a_{21} + a_{12}a_{22}}{\sqrt{(a_{11}^2 + a_{12}^2)(a_{21}^2 + a_{22}^2)}} = \rho
\end{aligned}
$$

   So to generate a bivariate normal with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation ρ, you would need to pick values $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ that yield the desired values. This can be done in many ways (since there are 5 values and 6 unknowns). One possible way is by

$$
\begin{aligned}
b_1 &= \mu_1 \\
b_2 &= \mu_2 \\
a_{11} &= \sigma_1 \\
a_{12} &= 0 \\
a_{21} &= \rho\sigma_2 \\
a_{22} &= \sigma_2\sqrt{1-\rho^2}
\end{aligned}
$$

3. Rice 3.59

   As discussed in class, the density of the minimum n iid RVs with density $f_T(t)$ is given by

$$
nf_T(v)[1 - F_T(v)]^{n-1}
$$

   For the given Weibull density, the CDF is

$$
F_T(t) = 1 - e^{-(t/\alpha)^\beta}
$$

   So plugging into the formula gives

$$
\begin{aligned}
f_V(v) &= \frac{n\beta}{\alpha^\beta} v^{\beta-1} e^{-(v/\alpha)^\beta} \left(e^{-(v/\alpha)^\beta}\right)^{n-1} \\
&= \frac{n\beta}{\alpha^\beta} v^{\beta-1} e^{-(v/\alpha)^{n\beta}}
\end{aligned}
$$

4. Rice 4.52

(a)

$$\begin{aligned}
E[Z] &= \frac{1}{h}\left(E[f(x+h)+\epsilon_2] - E[f(x)+\epsilon_1]\right) \\
&= \frac{f(x+h)-f(x)}{h} + \frac{1}{h}\left(E[\epsilon_2] - E[\epsilon_1]\right) \\
&= \frac{f(x+h)-f(x)}{h}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Z) &= \frac{1}{h^2}\left(\mathrm{Var}(f(x+h)+\epsilon_2) + \mathrm{Var}(f(x)+\epsilon_1)\right) \\
&= \frac{1}{h^2}\left(\mathrm{Var}(\epsilon_2) + \mathrm{Var}(\epsilon_1)\right) \\
&= \frac{2\sigma^2}{h^2}
\end{aligned}$$

As $h \to 0$, $E[Z] \to f'(x)$ but $\mathrm{Var}(Z) \to \infty$.

(b)

$$\begin{aligned}
MSE_h(Z) &= E[(Z - f'(x))^2] \\
&= \mathrm{Var}(Z) + (E[Z] - f'(x))^2 \\
&= \frac{2\sigma^2}{h^2} + \left(\frac{f(x+h)-f(x)}{h} - f'(x)\right)^2 \\
&\approx \frac{2\sigma^2}{h^2} + \left(\frac{f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 - f(x)}{h} - f'(x)\right)^2 \\
&= \frac{2\sigma^2}{h^2} + \left(\frac{f'(x)h + \frac{1}{2}f''(x)h^2}{h} - f'(x)\right)^2 \\
&= \frac{2\sigma^2}{h^2} + \frac{1}{4}(f''(x))^2h^2
\end{aligned}$$

$$\frac{d}{dh}MSE_h(Z) \approx \frac{-\sigma^2}{h^3} + \frac{1}{2}(f''(x))^2h$$

Setting this to 0 and solving for h gives

$$h_{opt} = \left(\frac{2\sigma^2}{(f''(x))^2}\right)^{1/4}$$

(c) Let the 3 measured points be

$$X_1 = f(x-h) + \epsilon_1$$
$$X_2 = f(x) + \epsilon_2$$
$$X_3 = f(x+h) + \epsilon_3$$

Then $f''(x)$ can be estimated by

$$U = \frac{X_3 - 2X_2 + X_1}{h^2}$$

Then

$$\begin{aligned} E[U] &= \frac{1}{h^2} E[f(x+h) + \epsilon_3] - 2E[f(x) + \epsilon_2] + E[f(x-h) + \epsilon_1] \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \end{aligned}$$

$$\begin{aligned} \text{Var}(U) &= \frac{1}{h^4} \text{Var}(f(x+h) + \epsilon_3) - 4\text{Var}(f(x) + \epsilon_2) + \text{Var}(f(x-h) + \epsilon_1) \\ &= \frac{\sigma^2 + 4\sigma^2 + \sigma^2}{h^4} = \frac{6\sigma^2}{h^4} \end{aligned}$$

To get a handle on the bias (not required in the problem), note

$$f(x+h) \approx f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(4)}(x)h^4$$
$$f(x-h) \approx f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(4)}(x)h^4$$

Then the bias of this estimate is given by

$$\begin{aligned} Bias(U) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \\ &\approx \frac{\frac{1}{12}f^{(4)}(x)h^4}{h^2} \\ &= \frac{1}{12}f^{(4)}(x)h^2 \end{aligned}$$

The optimal choose of h for this problem is

$$h_{opt} = \left(\frac{144\sigma^2}{f^{(4)}(x)}\right)^{1/6}$$

5. Rice 4.53

$$
\begin{aligned}
E[X] &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x dx dy \\
&= \int_{-1}^{1} x^2 |_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\
&= \int_{-1}^{1} 0 dy \\
&= 0
\end{aligned}
$$

Note that this is similar to $E[X] = E[E[X|Y]]$ approach. If the order of integration is switched, it is more difficult.

$$
\begin{aligned}
E[X] &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dy dx \\
&= \int_{-1}^{1} xy|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\
&= \int_{-1}^{1} 2x\sqrt{1-x^2} dx \\
&= 0
\end{aligned}
$$

If you were to use the $E[E[X|Y]]$ approach, note that

$$
f_{X|Y}(x,y) = 2\sqrt{1-y^2}; \qquad -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}
$$

which implies $E[X|Y=y] = 0$ for all y. Also note, that since this density depends on y, X and Y are not independent.

By symmetry, $E[Y] = 0$. This implies that

$$
\begin{aligned}
\text{Cov}(X,Y) &= E[XY] - E[X]E[Y] \\
&= E[XY] \\
&= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy dx dy \\
&= \int_{-1}^{1} y \left(x^2 |_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \right) dy \\
&= \int_{-1}^{1} y \times 0 dy \\
&= 0
\end{aligned}
$$

This can also be calculated by the iterated expectation approach as

$$
\begin{aligned}
E[XY] &= E[E[XY|Y]] \\
&= E[YE[X|Y]] \\
&= E[Y \times 0] \\
&= 0
\end{aligned}
$$