Statistics 110 – Assignment 5 Solutions **Summer, 2006**

1. Rice 4.38

   Using Chebyshev's Inequality

$$P[|X - E(X)| > k\sigma] \leq \frac{1}{k^2}; \qquad \text{for } k = 2, 3, 4$$

$$\frac{1}{k^2} = \frac{1}{4} = 0.25$$
$$\frac{1}{9} = 0.1111$$
$$\frac{1}{16} = 0.0625$$

   Since $X \sim Exp(\frac{1}{\sigma})$, $E(X) = \mathrm{SD}(X) = \sigma$ and $X > 0$,

$$\begin{aligned} P[|X - E(X)| > k\sigma] &= P[|X - \sigma| > k\sigma] \\ &= P[X > (k+1)\sigma] + P[X < -(k-1)\sigma] \\ &= P[X > (k+1)\sigma] \qquad \text{if } k \geq 1 \\ &= e^{-(k+1)\sigma/\sigma} = e^{-(k+1)} \end{aligned}$$

   Then for $k = 2$, $e^{-3} = 0.04978707$;
   $k = 3$, $e^{-4} = 0.01831564$;
   $k = 4$, $e^{-5} = 0.006737947$;

2. Rice 4.68

   Suppose there are X offsprings in the second generation, and $E(X) = \mu, \mathrm{Var}(X) = \sigma^2$. For each one of these X second generation offsprings, its offspring number is $Y_i, i = 1, 2, \ldots, X$. So the total number of the third generation offsprings is:

$$Z = Y_1 + Y_2 + ... + Y_X$$

$$\begin{aligned} E(Z) &= E[Y_1 + Y_2 + ... + Y_X] \\ &= E[E[Y_1 + Y_2 + ... + Y_X|X]] \\ &= E[X\mu] = \mu E[X] = \mu^2; \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Z) &= \mathrm{Var}(Y_1 + Y_2 + ... + Y_X) \\ &= E[\mathrm{Var}(Y_1 + Y_2 + ... + Y_X|X)] + \mathrm{Var}(E[Y_1 + Y_2 + ... + Y_X|X]) \\ &= E[X\sigma^2] + \mathrm{Var}[X\mu] \\ &= \mu\sigma^2 + \mu^2\sigma^2 \\ &= \sigma^2\mu(1 + \mu). \end{aligned}$$

3. Rice 4.70

   $f_{X,Y}(x,y) = \frac{2}{\pi}$ for $0 \le y \le 1$ and $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$. Then

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{\int_X f(x,y)dx} \\ &= \frac{\frac{2}{\pi}}{\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}dx} \\ &= \frac{1}{2\sqrt{1-y^2}} \end{aligned}$$

   Similarly

$$f_{Y|X}(y|x) = \frac{\frac{2}{\pi}}{\int_0^{\sqrt{1-x^2}} \frac{2}{\pi}dy} = \frac{1}{\sqrt{1-x^2}}.$$

   Note that each of conditional distributions is uniform over the appropriate range.

   So if given X, the best prediction for Y with minimum mean square error is the conditional mean of Y:

$$E(Y|X) = \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}} y dy = \frac{\sqrt{1-x^2}}{2}.$$

   And given Y, the best prediction for X with minimum mean square error is the conditional mean of X:

$$E(X|Y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{2\sqrt{1-y^2}} x dx = 0.$$

4. Rice 4.74

   First note that

$$E[X] = \int_0^1 2x^2 dx = \frac{2}{3}$$

$$E[X^2] = \int_0^1 2x^3 dx = \frac{1}{2}$$

$$
\begin{aligned}
M_X(t) &= \int_0^1 e^{tx} 2x dx \\
&= \left.\frac{2xe^{tx}}{t}\right|_0^1 - \int_0^1 \frac{2e^{tx}}{t} dx \\
&= \frac{2e^t}{t} - \left.\frac{2e^{tx}}{t^2}\right|_0^1 \\
&= \frac{2e^t}{t} + \frac{2}{t^2} - \frac{2e^t}{t^2} \\
&= 2\frac{te^t - e^t + 1}{t^2}
\end{aligned}
$$

$$
\begin{aligned}
M'(t) &= 2\frac{te^t + e^t - e^t}{t^2} - 4\frac{te^t - e^t + 1}{t^3} \\
&= 2\frac{t^2e^t - 2te^t + 2e^t - 2}{t^3}
\end{aligned}
$$

$$
\begin{aligned}
M''(t) &= 2\frac{t^2e^t + 2te^t - 2te^t - 2e^t + 2e^t}{t^3} - 3\frac{t^2e^t - 2te^t + 2e^t - 2}{t^4} \\
&= 2\frac{t^3e^t - 3t^2e^t + 6te^t - 6e^t + 6}{t^4}
\end{aligned}
$$

While none of these are strictly defined at 0, they do have limits as $t \to 0$, which can be calculated by L'Hopital's rule

$$
\begin{aligned}
\lim_{t\to 0} M'(t) &= \lim_{t\to 0} 2\frac{2te^t + t^2e^t - 2e^t - 2te^t + 2e^t}{3t^2} \\
&= \lim_{t\to 0} 2\frac{t^2e^t}{3t^2} \\
&= \lim_{t\to 0} \frac{2}{3}e^t = \frac{2}{3} = E[X]
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t\to 0} M''(t) &= \lim_{t\to 0} 2\frac{t^3e^t + 3t^2e^t - 3t^2e^t - 6te^t + 6te^t + 6e^t - 6e^t}{4t^3} \\
&= \lim_{t\to 0} \frac{t^3e^t}{2t^3} \\
&= \lim_{t\to 0} \frac{e^t}{2} = \frac{1}{2} = E[X^2]
\end{aligned}
$$

5. Rice 4.76

   Let $Y = \sum_{i=1}^{n} X_i$ and $X_i \overset{iid}{\sim} Bin(n,p)$. Then

$$M_Y(t) = [pe^t + (1-p)]^n;$$
$$M'_Y(t) = n[pe^t + (1-p)]^{n-1}pe^t;$$
$$M''_Y(t) = n(n-1)[pe^t + (1-p)]^{n-2}p^2e^{2t} + n[pe^t + (1-p)]^{n-1}pe^t;$$

   then

$$E(Y) = M'_Y(t)|_{t=0} = np,$$

   and

$$\text{Var}(X) = M''_Y(t)|_{t=0} - E^2(Y) = n(n-1)p^2 + np - n^2p^2 = np(1-p).$$

6. Note as shown in question 4, $E[p] = \frac{2}{3}$ and $E[p^2] = \frac{1}{2}$

$$\begin{aligned} M_X(t) &= E[M_{X|p}(t)] \\ &= E[(1 + p(e^t - 1))^2] \\ &= E[1 + 2p(e^t - 1) + p^2(e^t - 1)^2] \\ &= 1 + 2(e^t - 1)E[p] + (e^t - 1)^2E[p^2] \\ &= 1 + (e^t - 1)\frac{4}{3} + (e^t - 1)^2\frac{1}{2} \\ &= \left(1 - \frac{4}{3} + \frac{1}{2}\right) + \left(\frac{4}{3} - 1\right)e^t + \frac{e^{2t}}{2} \\ &= \frac{1}{6} + \frac{e^t}{3} + \frac{e^{2t}}{2} = \frac{1 + 2e^t + 3e^{2t}}{6} \end{aligned}$$

7. Rice 6.4

$$P[|T| < t_0] = P[-t_0 < T < t_0]$$

   Due to the symmetry of the t-distribution,

$$P[T > t_0] = P[T_0 < -t_0] = 0.05$$

   which implies

$$P[-t_0 < T < t_0] = 0.9$$

   so for both (a) and (b) $t_0 = 1.895$.

8. Rice 6.8

   For $X \sim Exp(1)$, $2X \sim Exp(\frac{1}{2})$.

   For $v \sim \chi^2(2)$, $f(v) = \frac{1}{2}e^{-\frac{v}{2}}$. So $2X \sim \chi^2(2)$. And for $X, Y \overset{iid}{\sim} exp(1)$,

$$\frac{X}{Y} = \frac{2X}{2Y} \sim F_{2,2}$$

9.

$$\bar{X} \sim N\left(0, \frac{1}{25}\right)$$

$$\begin{aligned} P[|\bar{X}| < c] &= P[|5\bar{X}| < 5c] \\ &= 2\Phi(5c) - 1 = 0.5 \end{aligned}$$

So $\Phi(5c) = 0.75$, which implies $5c = 0.674$ or $c = 0.1348$.

If we can't assume normality, by Chebyshev's inequality,

$$\begin{aligned} P[|\bar{X}| \leq 0.1348] &\geq 1 - \frac{\sigma^2}{0.1348^2} \\ &= 1 - \frac{1}{25 \times 0.01817} = -1.20 \end{aligned}$$

Thus all we can say in this case is that $P[|\bar{X}| \leq 0.1348] \geq 0$.

10. Rice 6.10

Since $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$,

$$\begin{aligned} P[a < S^2/\sigma^2 < b] &= P[(n-1)a < (n-1)S^2/\sigma^2 < (n-1)b] \\ &= F_{\chi^2_{n-1}}((n-1)b) - F_{\chi^2_{n-1}}((n-1)a) \end{aligned}$$

11. $f(x) = x^2$ is a convex function, since

$$\begin{aligned} (\lambda x_1 + (1-\lambda)x_2)^2 - (\lambda x_1^2 + (1-\lambda)x_2^2) &= \lambda(1-\lambda)x_1^2 + \lambda(1-\lambda)x_2^2 + 2\lambda(1-\lambda)x_1x_2 \\ &= \lambda(1-\lambda)(x_1^2 + 2x_1x_2 + x_2^2) \\ &= \lambda(1-\lambda)(x_1 + x_2)^2 \geq 0 \\ &\Rightarrow \lambda x_1^2 + (1-\lambda)x_2^2 \geq (\lambda x_1 + (1-\lambda)x_2)^2 \end{aligned}$$

Or you can show this by $f''(x) = 2 > 0$ for all x.

So by Jensen's inequality

$$(E[S])^2 \leq E[S^2] \qquad \Rightarrow \qquad E[S] \leq \sqrt{E[S^2]}$$

12. Note that $P[X \leq 26] = 1 - P[X \geq 27]$ and $E[X] = \mathrm{Var}(X) = 20$ and the moment generating function is $M_X(t) = \exp(20(e^t - 1))$.

(a) By the Markov inequality

$$P[X \geq 27] \leq \frac{20}{27} \Rightarrow P[X \leq 26] \geq 1 - \frac{20}{27} = 0.259$$

(b)

$$\begin{aligned}P[X \geq 27] &= P[X \geq 20 + 7] \\ &\leq \frac{20}{20 + 7^2} = \frac{20}{69} \\ &\Rightarrow P[X \leq 26] \geq 1 - \frac{20}{69} = 0.710\end{aligned}$$

(c) To get a lower bound on p, for $t > 0$

$$P[X \geq 27] \leq e^{-27t} \exp(20(e^t - 1))$$

To maximize this lower bound, we need to minimize

$$20(e^t - 1) - 26t$$

This is minimized by setting $e^t = \frac{27}{20}$ as

$$\frac{d}{dt} 20(e^t - 1) - 26t = 20e^t - 26 < 0$$

This gives

$$\begin{aligned}P[X \geq 27] &\leq \exp(20(27/20 - 1)) \left(\frac{20}{27}\right)^{27} \\ &= \frac{e^{-20}(20e)^{27}}{27^{27}}\end{aligned}$$

$$= 0.3319$$

Thus

$$P[X \leq 26] \geq 1 - 0.3319 = 0.6681$$

To get a upper bound on p, for $t < 0$

$$P[X \leq 26] \leq e^{-26t} \exp(20(e^t - 1))$$

To minimize this upper bound, we need to minimize

$$20(e^t - 1) - 26t$$

This is minimized by setting $t = 0$ as

$$\frac{d}{dt} 20(e^t - 1) - 26t = 20e^t - 26 < 0$$

This gives (sort of since $t < 0$ for this bound)

$$P[X \leq 26] \leq e^0 \exp(20(e^0 - 1)) = 1$$

Suggested Problems

1. Rice 4.60

$$
\begin{aligned}
E[X] &= E[E[X|\text{speed}] \\
&= 1 \times \frac{2}{3} + 3\frac{1}{3} \\
&= \frac{5}{3} = 1.67
\end{aligned}
$$

2. Rice 4.64
   If independent $f_{X|Y}(x|y) = f_X(x)$. Thus

$$
E[X|Y=y] = \int x f_{X|Y}(x|y)dx = \int x f_X(x)dx = E[X]
$$

   (Similar proof for the discrete case.)

3. Rice 4.66

$$
\begin{aligned}
E[\text{items}] &= pE[\text{items}|\text{present}] + (1-p)E[\text{items}|\text{not present}] \\
&= p\frac{n+1}{2} + (1-p)n \\
&= \frac{2n - np + 1}{2} = \frac{n(2-p)+1}{2}
\end{aligned}
$$

4. Rice 4.67 Let Y be the number of heads in the second set of flips. Thus the total number of heads is $Z = N + Y$. Then

$$
\begin{aligned}
E[N+Y] &= E[E[N+Y|N]] \\
&= E[N + E[Y|N]] \\
&= E\left[N + \frac{N}{2}\right] = \frac{3}{2}E[N] \\
&= \frac{3}{2}\frac{n}{2} = \frac{3}{4}n
\end{aligned}
$$

5. Rice 4.90

$$
E[XY] = \frac{\partial^2}{\partial s \partial t} M_{X,Y}(s,t)|_{s=0,t=0}
$$

6. Rice 4.91

$$\begin{aligned} M_{aX+bY}(t) &= M_{aX}(t)M_{bY}(t) \\ &= M_X(at)M_Y(bt) \end{aligned}$$

$$M'_{aX+bY}(t) = aM'_X(at)M_Y(bt) + bM_X(at)M'_Y(bt)$$

$$M'_{aX+bY}(0) = aE[X] + bE[Y]$$

$$M''_{aX+bY}(t) = a^2M''_X(at)M_Y(bt) + 2abM'_X(at)M'_Y(bt) + b^2M_X(at)M''_Y(bt)$$

$$M''_{aX+bY}(0) = a^2E[X^2] + 2abE[X]E[Y] + b^2E[Y^2]$$

$$\begin{aligned} \mathrm{Var}(aX + bY) &= (a^2E[X^2] + 2abE[X]E[Y] + b^2E[Y^2]) - (aE[X] + bE[Y])^2 \\ &= a^2(E[X^2] - (E[X])^2) + b^2(E[Y^2] - (E[Y])^2) \\ &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) \end{aligned}$$