Statistics 110 – Assignment 6 Solutions **Summer, 2006**

1. Rice 4.94

$$E\left[\frac{1}{X}\right] = \int_{10}^{20} \frac{1}{10}\frac{1}{x}dx = \frac{\ln 2}{10} = 0.06931472$$
$$E\left[\frac{1}{X^2}\right] = \int_{10}^{20} \frac{1}{10}\frac{1}{x^2}dx = \frac{1}{200} = 0.005$$
$$\text{Var}\left(\frac{1}{X}\right) = 0.005 - 0.06931472^2 = 0.00019751$$

Based on $E(X) = 15$, $\text{Var}(X) = \frac{25}{3}$. Let $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$, and $f''(x) = \frac{2}{x^3}$ so

$$f(x) \approx \frac{1}{E[X]} - \frac{1}{(E[X])^2}(x - E[X])$$

and

$$Approx\left(E\left[\frac{1}{X}\right]\right) = \frac{1}{E[X]} = \frac{1}{15} = 0.06667$$
$$Approx\left(\text{Var}\left(\frac{1}{X}\right)\right) = \frac{1}{(E[X])^4}\text{Var}(X) = \frac{1}{15^4}\frac{25}{3} = 0.000165$$

The second order approximation to the mean is

$$Approx\left(E\left[\frac{1}{X}\right]\right) = \frac{1}{E[X]} + \frac{2}{(E[X])^3}\text{Var}(X) = \frac{1}{15} + \frac{1}{15^3}\frac{25}{3} = 0.06914$$

2. Rice 4.98

(a) Let $E[R] = \mu_R, \text{Var}(R) = \sigma_R^2, E[\theta] = \mu_\theta, \text{Var}(\theta) = \sigma_\theta^2$. Then for $g(R, \theta) = R \sin\theta$

$$Y \approx \mu_R \sin\mu_\theta + \sin\mu_\theta(R - \mu_R) + R\cos\mu_\theta(\theta - \mu_\theta)$$

$$Approx(\text{Var}(Y)) = \sigma_R^2 \sin^2\mu_\theta + \mu_R^2\sigma_\theta^2\cos^2\mu_\theta$$

(b) if known $R = r$, then $Y = r\sin\theta$ and

$$\text{Var}(Y) \approx r^2\cos^2 E(\theta)Var(\theta)$$

$\theta = 0$ leads to the most variable estimated altitude as

$$\frac{d}{d\theta}r^2\sigma_\theta^2\cos^2\theta = 2r^2\sigma_\theta^2\sin\theta\cos\theta$$

takes the value 0 for $\theta = 0$ and $\frac{\pi}{2}$. By checking the function, $\theta = 0$ corresponds to the maximum and $\frac{\pi}{2}$ to a minimum.
This answer is also supported by noting that

$$\frac{dY}{d\theta} = r\cos\theta$$

is maximized by $\theta = 0$.

3. Rice 5.5
Since $X_n \sim Bin(n, p)$,

$$M_{X_n}(t) = (1 + p(e^t - 1))^n$$

Making the substitution $p = \frac{\lambda}{n}$

$$M_{X_n}(t) = \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n$$

Then as $(1 + \frac{a}{n})^n \to e^a$,

$$M_{X_n}(t) \to \exp(\lambda(e^t - 1))$$

which is the moment generating function of a $Pois(\lambda)$ random variable, implying $X_n \xrightarrow{\mathcal{D}} Pois(\lambda)$.

4. Rice 5.12
Let

$$X_i \sim Unif(-0.5, 0.5) \qquad E[X_i] = 0 \qquad Var(X_i) = \frac{1}{12}$$

So

$$\frac{\sum_{i=1}^{100} X_i - 0}{\sqrt{\frac{1}{12}100}} \overset{approx.}{\sim} N(0,1)$$

implying

$$P\left[\left|\sum_{i=1}^{100} X_i\right| > 1\right] \approx P\left[\left|\frac{\sum_{i=1}^{100} X_i}{\sqrt{\frac{100}{12}}}\right| > \frac{\sqrt{3}}{5}\right]$$

$$= 2\left(1 - \Phi\left(\frac{\sqrt{3}}{5}\right)\right) = 0.73$$

Similarly

$$P\left[\left|\sum_{i=1}^{100} X_i\right| > 2\right] \approx 2\left(1 - \Phi\left(2\frac{\sqrt{3}}{5}\right)\right) = 0.4884224$$

and

$$P\left[\left|\sum_{i=1}^{100} X_i\right| > 5\right] \approx 2\left(1 - \Phi\left(\sqrt{3}\right)\right) = 0.08326452$$

5. Rice 5.14

The result of step i follows the distribution $P[X_i = +50] = \frac{2}{3}$ (North) and $P[X_i = -50] = \frac{1}{3}$ (South). Then

$$\begin{aligned} E[X_i] &= 50\frac{2}{3} - 50\frac{1}{3} = \frac{50}{3} \\ \text{Var}(X_i) &= 50^2\frac{2}{3} + (-50)^2\frac{1}{3} - \left(\frac{50}{3}\right) \\ &= \frac{2500}{3} - \frac{2500}{9} = \frac{5000}{9} \end{aligned}$$

The final position is given by $S_{60} = \sum_{i=1}^{60} X_i$. Then

$$\begin{aligned} E[S_{60}] &= 60\frac{50}{3} = 1000 \\ \text{Var}(S_{60}) &= 60\frac{5000}{9} = 33333.3 \end{aligned}$$

So $S_{60} \overset{approx.}{\sim} N(1000, 33333.3)$. So the most likely position is 1000cm (= 10 m) to the north of his starting position. The standard deviation of the difference from this is 182.6 cm.

6. Rice 5.23

Let $Z_1, \ldots, Z_n$ be a set of iid indicator variables. Then $P[Z_i = 1] = A$ and $P[Z_i = 0] = 1 - A$, so $E[Z_i] = A$. Then by the weak law of large numbers, $\bar{Z} \xrightarrow{P} E[Z_i] = A$.

7. Rice 5.24

Note that $\sum_{i=1}^{n} Z_i \sim Bin(n, A)$. So as $n \to \infty$, $\hat{A} \overset{approx.}{\sim} N\left(A, \frac{A(1-A)}{n}\right)$. So we need to choose n such that

$$\begin{aligned} 0.99 &= P[|\hat{A} - A| < 0.01] \\ &= P\left[\left|\frac{(\hat{A} - A)\sqrt{n}}{\sqrt{A(1-A)}}\right| < \frac{0.01\sqrt{n}}{\sqrt{A(1-A)}}\right] \\ &\approx P\left[|Z| < \frac{0.01\sqrt{n}}{\sqrt{A(1-A)}}\right] \end{aligned}$$

This implies that (with $A = 0.2$)

$$\frac{0.01\sqrt{n}}{\sqrt{0.16}} = 2.576$$

or

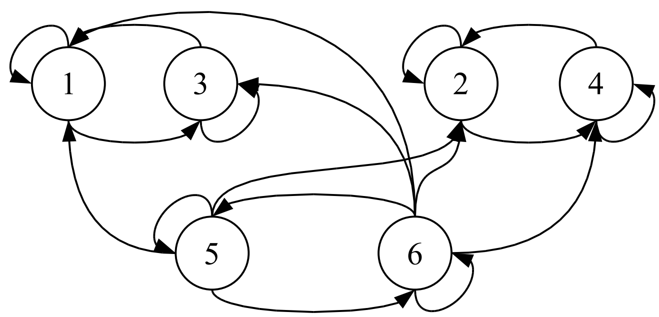
$$n = \frac{2.576^2 0.16}{0.01^2} = 10617.2$$

8. $nF_n(x) \sim Bin(n, F(x))$, so by the weak law of law numbers, $F_n(x)$ converges in probability its success probability $F(x)$.

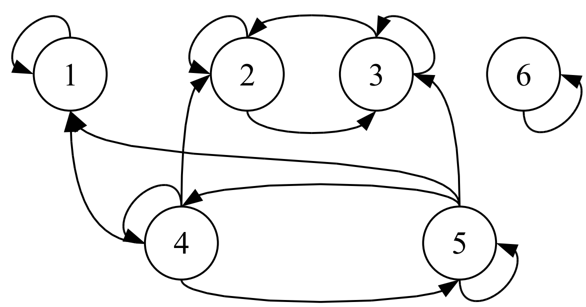
   Another approach, is to note that $E[F_n(x)] = F(x)$ and $\text{Var}(F_n(x)) = \frac{F(x)(1-F(x))}{n}$. By Chebyshev's inequality,

$$P[|F_n(x) - F(x)| \geq \delta] \leq \frac{F(x)(1 - F(x))}{n\delta^2} \to 0$$

9. (a) (1,3), (2,4), (5,6) are communicating classes, with (1,3), (2,4) being recurrent, and (5,6) being transient.



   (b) (1), (2,3),(6) are communicating recurrent classes, and (4,5) is a communicating transient class.



10. Let states 1,2,3 are notations for states C, S, G respectively. Then the transition matrix

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

To find stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is to solve

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

subject to $\pi_1 + \pi_2 + \pi_3 = 1$. The solution is

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3) = \left(\frac{30}{59}, \frac{16}{59}, \frac{13}{59}\right)$$

The the proportion of days that Buffy cheerful is $\pi_1 = \frac{30}{59}$ and the long-run average number of iterations between gloomy days is $\frac{1}{\pi_3} = \frac{59}{13} = 4.54$ day by applying the result that

$$\pi_3 = \frac{1}{E_3(T_y)}$$

11. There are three possible states in this problem: state 1 is that both switches are off, state 2 is that one is off and the other is on, and state 3 is that both switches are on. Then the transition probability matrix is

$$P = \begin{bmatrix} \frac{3}{4} \times \frac{3}{4} & 2\frac{3}{4} \times \frac{1}{4} & \frac{1}{4} \times \frac{1}{4} \\ \frac{1}{2} \times \frac{1}{2} & 2\frac{1}{2} \times \frac{1}{2} & \frac{1}{2} \times \frac{1}{2} \\ \frac{1}{4} \times \frac{1}{4} & 2\frac{3}{4} \times \frac{1}{4} & \frac{3}{4} \times \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\ \frac{4}{16} & \frac{8}{16} & \frac{4}{16} \\ \frac{1}{16} & \frac{6}{16} & \frac{9}{16} \end{bmatrix}$$

Solving the equations

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

with $\pi_1 + \pi_2 + \pi_3 = 1$, then we get

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3) = \left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

So the fraction of time that both switches are on is $\frac{2}{7}$.

Suggested Problems

1. Rice 5.13

The result of step i follows the distribution $P[X_i = +50] = \frac{1}{2}$ (North) and $P[X_i = -50] = \frac{1}{2}$ (South). Then

$$E[X_i] = 50\frac{1}{2} - 5012 = 0$$
$$\text{Var}(X_i) = 50^2\frac{1}{2} + (-50)^2\frac{1}{2} - 0 = 2500$$

The final position is given by $S_{60} = \sum_{i=1}^{60} X_i$. Then

$$E[S_{60}] = 60 \times 0 = 0$$
$$\text{Var}(S_{60}) = 60 \times 2500 = 150,000$$

So $S_{60} \overset{approx.}{\sim} N(0, 150000)$. So the most likely position is the initial starting position. The standard deviation of the difference from this is 387.3 cm.

2. Rice 5.17

$$P[|\bar{X}-\mu|<1] = P\left[\left|\frac{\sqrt{n}(\bar{X}-\mu)}{5}\right| < \frac{\sqrt{n}}{5}\right]$$
$$\approx P\left[|Z| < \frac{\sqrt{n}}{5}\right] = 0.95$$

So need n such that

$$\frac{\sqrt{n}}{5} = 1.96 \qquad \Rightarrow \qquad n = (5 \times 1.96)^2 = 96.04$$

The Chebyshev approach gives

$$P[|\bar{X}-\mu|<1] \geq 1 - \frac{\mathrm{Var}(\bar{X})}{1^2} = 1 - \mathrm{Var}(\bar{X})$$

Thus we need to solve

$$\mathrm{Var}(\bar{X}) = \frac{25}{n} = 0.05$$

which yields $n = \frac{25}{0.05} = 500$ as a upper bound for n.

3. Rice 5.26

$$S \sim Bin(25, 0.3) \approx N(7.5, 5.25)$$

Let $Y \sim N(7.5, 5.25)$

x	Exact $P[S \leq x]$	Normal Approximation $P[Y \leq x]$	Corrected Approximation $P[Y \leq x + 0.5]$
5	0.1935	0.1376	0.1913
7	0.5118	0.4136	0.5000
9	0.8106	0.7437	0.8086
11	0.9558	0.9367	0.9596