Distributions Related to the Normal

Statistics 110

Summer 2006
Distributions Related to the Normal

The are a number of very common distributions related to the normal distribution. These distributions underlie many common inference procedures in statistics.

• Chi-square distributions ($\chi^2_{df}$)

**Definition.** If $Z \sim N(0, 1)$, then $U = Z^2$ has a Chi-square distribution with 1 degree of freedom ($\chi^2_1$).

The CDF of the of $U$ is given by

$$F_U(u) = P[U \leq u] = P[-\sqrt{u} \leq Z \leq \sqrt{u}] = \Phi(\sqrt{u}) - \Phi(-\sqrt{u})$$
so the density is given by

\[ f_U(u) = \frac{d}{du} \left( \Phi(\sqrt{u}) - \Phi(-\sqrt{u}) \right) \]

\[ = \frac{1}{2\sqrt{u}} \phi(\sqrt{u}) + \frac{1}{2\sqrt{u}} \phi(-\sqrt{u}) \]

\[ = \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \]

\[ = \frac{u^{-1/2}}{2^{1/2} \sqrt{\pi}} e^{-u/2}; \quad u \geq 0 \]

Note that this happens to be the same as a \textit{Gamma} \( \left( \frac{1}{2}, \frac{1}{2} \right) \) distribution.

Note that if \( X \sim N(\mu, \sigma^2) \), then

\[ \frac{(X - \mu)}{\sigma} \sim N(0, 1) \]

so

\[ \left[ \frac{(X - \mu)}{\sigma} \right]^2 \sim \chi_1^2 \]
Definition. If $U_1, U_2, \ldots, U_n$ are independent $\chi^2_1$ RVs, then the distribution of $V = U_1 + U_2 + \ldots + U_n$ is a Chi-square distribution with $n$ degrees of freedom ($\chi^2_n$).

Note that this is the same as a $Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$, so the density is

$$f_V(v) = \frac{v^{(n/2)-1}}{2^{n/2} \Gamma(n/2)} e^{-v/2}; \quad v \geq 0$$

and $E(V) = n, \ Var(V) = 2n$. Also the MGF is

$$M_V(t) = \frac{1}{(1 - 2t)^{n/2}}$$

Since chi-squared random variables are special cases of gamma ($\lambda = \frac{1}{2}$), if $U \sim \chi^2_n$ is independent of $V \sim \chi^2_m$, then $U + V \sim \chi^2_{m+n}$. 
Chi-squared distributions

Distributions Related to the Normal
• $t$ distributions ($t_{df}$)

**Definition.** If $Z \sim N(0, 1)$ and $U \sim \chi^2_n$, and $Z$ and $U$ are independent, then the distribution of

$$t = \frac{Z}{\sqrt{U/n}}$$

*is called the $t$ distribution with $n$ degrees of freedom ($t_n$).*

The density of the $t_n$ distribution is

$$f_t(t) = \frac{\Gamma((n + 1)/2)}{\sqrt{n\pi} \Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$
Note that the $t_1$ distribution is the same as the Cauchy(0,1) distribution.
$E[t^r] < \infty$ only if $r < n$. So $t_1$ has no moments, $t_2$ has a mean but no variance. Every other $t_n$ distribution with $n > 2$ has a mean and a variance with

$$E[t] = 0; \quad \text{Var}(t) = \frac{n}{n - 2}$$

Since the $t_n$ distributions don’t have all moments, there is no MGF.

As can be easily seen, $t_n$ distributions are symmetric about 0. Also their variance is greater than 1, but decreasing towards 1 as the degrees of freedom increases.

Also as $n \to \infty$, the $t_n$ distribution converges to the $N(0, 1)$ distribution. In many cases, t-tables, such as the one in Rice, denote the standard normal distribution as $t_\infty$. 
• \( F \) distributions \((F_{df_1,df_2})\)

**Definition.** Let \( U \) and \( V \) be independent with chi-squared RVs with \( m \) and \( n \) degrees of freedom respectively. The distribution of

\[
W = \frac{U/m}{V/n}
\]

is called the \( F \) distribution with \( m \) and \( n \) degrees of freedom \((F_{m,n})\).

The density of the \( F_{m,n} \) distribution is

\[
f_F(w) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n}w\right)^{-(m+n)/2} ; \quad w \geq 0
\]

\[
E[W] = \frac{n}{n-2} \text{ for } n > 2; \quad \text{Var}(W) = \frac{2n^2(m + n - 2)}{m(n - 2)^2(n - 4)} \text{ for } n > 4
\]
Note that if $W \sim F_{m,n}$, then $1/W \sim F_{n,m}$. Also the square of a $t_n$ RV has an $F_{1,n}$ distribution.
Suppose that \( X_1, X_2, \ldots, X_n \) are independent \( N(\mu, \sigma^2) \) RVs.

- What is the distribution of the sample mean

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

- What is the distribution of the sample variance

\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2
\]

- What is their joint distribution?
These are important if we want to understand how large a sample is needed to estimate the population mean, $\mu$, by the sample mean $\bar{X}$, to a given level of accuracy.

**Theorem.** Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ RVs. Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

**Proof.** As we have seen before, a linear combination of normals is normal. Therefore we only need to verify the moments.

\[
E[\bar{X}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} n \mu = \mu
\]

\[
\text{Var}(\bar{X}) = \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i) = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n}
\]

$\square$

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Uses of $t_n, \chi^2_n, F_{m,n}$
Definition. \( \mathbf{X} = \{X_1, \ldots, X_n\} \) has a multivariate normal \( N(\boldsymbol{\mu}, \Sigma) \) where \( \boldsymbol{\mu} = [\mu_1, \ldots, \mu_n]^T \) is the vector of means \( (\mu_i = E[X_i]) \) and

\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2
\end{bmatrix}
\]

is the Variance-Covariance matrix where \( \sigma_i^2 = \text{Var}(X_i) \) and \( \sigma_{ij} = \text{Cov}(X_i, X_j) \) with density

\[
f(\mathbf{x}) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)
\]

where \( \mathbf{x} = [x_1, \ldots, x_n]^T \).
Note the form inside the exponential is proportional to

\[(x - \mu)^T \Sigma^{-1} (x - \mu) \propto \]

\[\sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2} + 2 \sum_{i<j} \frac{c_{ij}(x_i - \mu_i)(x_j - \mu_j)}{\sigma_i \sigma_j} \]

**Lemma.** Let \( \{X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n\} \) be multivariate normal \( N(\mu, \Sigma) \) RVs. Then \( X = \{X_1, \ldots, X_m\} \) is independent of \( Y = \{Y_1, \ldots, Y_n\} \) if

\[\text{Cov}(X_i, Y_j) = 0 \text{ for all pairs } i, j\]

**Proof.** If \( \text{Cov}(X_i, Y_j) = 0 \), it can be shown that \( c_{ij} = 0 \) (due to the block structure of \( \Sigma \)), so the terms

\[\frac{c_{ij}(x_i - \mu_i)(y_j - \mu_j)}{\sigma_i \sigma_j}\]

drop out of the sum. Rearranging the remaining terms allows the joint
density to be factored as

\[ f(x, y) = g(x)h(y) \]

where \( g \) is the density of \( X \) and \( h \) is the density of \( Y \). Since the density factors this way, \( X \) is independent of \( Y \). \( \square \)

**Theorem.** Let \( X_1, X_2, \ldots, X_n \) be iid \( N(\mu, \sigma^2) \) RVs. Then \( \bar{X} \) is independent of \( \{(X_1 - \bar{X}), \ldots, (X_n - \bar{X})\} \)

**Proof.**

\[ X_1 - \bar{X} = \left(1 - \frac{1}{n}\right) X_1 - \frac{1}{n} X_2 - \ldots - \frac{1}{n} X_n \]

The other terms \( X_i - \bar{X} \) have an equivalent form.
\[
\text{Cov}(\bar{X}, X_1 - \bar{X}) = \text{Cov} \left( \frac{1}{n} \sum_{i=1}^{n} X_i, \left( 1 - \frac{1}{n} \right) X_1 - \frac{1}{n} \sum_{i=2}^{n} X_i \right) = \text{Cov} \left( \frac{1}{n} X_1, \left( 1 - \frac{1}{n} \right) X_1 \right) + \sum_{i=2}^{n} \text{Cov} \left( \frac{1}{n} X_i, -\frac{1}{n} X_i \right)
\]
\[
= \frac{1}{n} \left( 1 - \frac{1}{n} \right) \text{Var}(X_1) - \sum_{i=2}^{n} \frac{1}{n^2} \text{Var}(X_i)
\]
\[
= \frac{1}{n} \left( \frac{n-1}{n} \right) \sigma^2 - (n - 1) \frac{1}{n^2} \sigma^2 = 0
\]

Similarly $\text{Cov}(\bar{X}, X_i - \bar{X}) = 0$ for the other $i$. Since these covariances are all 0, the independence result holds. \(\square\)

**Corollary.** $\bar{X}$ and $S^2$ are independent.

This holds since if $X$ and $Y$ are independent, so are $g(X)$ and $h(Y)$.

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Uses of $t_n, \chi^2_n, F_{m,n}$
Theorem. \[ \frac{(n - 1)S^2}{\sigma^2} \sim \chi^2_{n-1} \]

Proof. First notice that

\[ \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2_n \]

Now

\[ \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} [(X_i - \bar{X}) + (\bar{X} - \mu)]^2 \]

If we expand the square, using the fact that \[ \sum_{i=1}^{n} (X_i - \bar{X}) = 0 \] we get
\[
\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2
\]

\[
= \frac{(n - 1)S^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2
\]

Now notice that this is of the form \( W = U + V \) and \( U \) and \( V \) are independent (since \( U \) is a function of \( S^2 \) and \( V \) is a function of \( \bar{X} \)).

So the MGFs satisfy \( M_W(t) = M_U(t)M_V(t) \). Since \( W \sim \chi^2_n \) and \( V \sim \chi^2_1 \), the MGF for \( U \) is

\[
M_U(t) = \frac{M_W(t)}{M_V(t)} = \frac{(1 - 2t)^{-n/2}}{(1 - 2t)^{-1/2}} = \left( \frac{1}{1 - 2t} \right)^{(n-1)/2}
\]

This is MGF for a \( \chi^2_{n-1} \) □
The term degrees of freedom come from the relationships such as \[ \sum_{i=1}^{n}(X_i - \bar{X}) = 0 \]. In this case, you can specify \( n - 1 \) of the \( \{X_i - \bar{X}\} \) to be whatever you want. However the remaining term is fixed \( ((X_n - \bar{X}) = - \sum_{i=1}^{n-1}(X_i - \bar{X})) \)

**Theorem.**

\[
E[\bar{X}] = \mu; \quad E[S^2] = \sigma^2
\]

(\( \bar{X} \) and \( S^2 \) are said to be unbiased estimates of \( \mu \) and \( \sigma^2 \)).

The first we’ve already proved and the second holds since

\[
S^2 = \frac{\sigma^2 \chi^2_{n-1}}{n-1}
\]

and \( E[\chi^2_{n-1}] = n - 1 \).

Also these estimators get more precise as \( n \) increases as

\[
\text{Var}(\bar{X}) = \frac{\sigma^2}{n}; \quad \text{Var}(S^2) = \frac{2\sigma^4}{n-1}
\]
If we want to perform inference on $\mu$ using $\bar{X}$ based on the results we have so far, we also need to know $\sigma^2$. It would be nice if we could do things without having to know this. In fact we can.

**Theorem.**

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Here we are standardizing by $S/\sqrt{n}$ instead of $\sigma/\sqrt{n}$.

**Proof.**

$$t = \left( \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right) \sqrt{S^2/\sigma^2}$$

The numerator has a standard normal distribution and the denominator is the square root of a $\chi^2_{n-1}$ RV divided by its degree of freedom. $\square$
While we can make statements about $\bar{X}$ without knowing $\sigma^2$, it does come at a cost.

As mentioned earlier, if $t \sim t_{n-1}$, $Z \sim N(0, 1)$, $\text{Var}(t) > \text{Var}(Z)$.

In addition $P[-c \leq t \leq c] < P[-c \leq Z \leq c]$ for all $c > 0$. Thus you need bigger intervals to cover the same probability with $ts$ than you do with normals.

We can expand some of these results to learn about difference between distributions. For example if $X_1, X_2, \ldots, X_m$ are iid $N(\mu_X, \sigma^2_X)$ and $Y_1, Y_2, \ldots, Y_n$ are iid $N(\mu_Y, \sigma^2_Y)$, then $\bar{X} - \bar{Y}$ is normally distributed with

$$E[\bar{X} - \bar{Y}] = \mu_X - \mu_Y; \quad \text{Var}(\bar{X} - \bar{Y}) = \frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}$$

Thus we can make probability statements about $\bar{X} - \bar{Y}$, an estimate of $\mu_X - \mu_Y$, assuming we know $\sigma^2_X$ and $\sigma^2_Y$. 

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Uses of $t_n, \chi^2_n, F_{m,n}$
One possible use of the final type of distribution mentioned, among many others, is to compare variances. Again, let $X_1, X_2, \ldots, X_m$ be iid $N(\mu_X, \sigma^2_X)$ and $Y_1, Y_2, \ldots, Y_n$ be iid $N(\mu_Y, \sigma^2_Y)$. If

$$S^2_X = \frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X})^2; \quad S^2_Y = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

then

$$\frac{S^2_X}{S^2_Y} \sim \frac{\sigma^2_X}{\sigma^2_Y} F_{m-1, n-1}$$

This distributional result allows us to make probability statements about $\frac{S^2_X}{S^2_Y}$, an estimate of variance ratio $\frac{\sigma^2_X}{\sigma^2_Y}$. 

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Uses of $t_n, \chi^2_n, F_{m,n}$
Making Probability Statements with $t_n, \chi^2_n, F_{m,n}$ Distributions

None of these distributions have nice CDF and quantile functions, so software or tables need to be used to make probability statements.

The tables that usually are presented are of the quantile function (as in done in the text). For example, for the $t_n$ distribution, the quantiles which satisfy

$$P[T \leq t_p] = p$$

for different choices of $p$ and $n$ are available.
Table 4: Percentiles of $t$ distribution

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<tr>
<th>$df$</th>
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<th>$t_{0.70}$</th>
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For example we can determine the value $c$ such that

$$P \left[ \left| \frac{\bar{X} - \mu}{s/\sqrt{n}} \right| \leq c \right] = 0.95$$

i.e. how many standard errors will your sample average be from the true mean 95% of the time.

This is equivalent to

$$P \left[ \frac{\bar{X} - \mu}{s/\sqrt{n}} \leq c \right] = 0.975$$

so $c = t_{0.975}$

For example if $n = 10$, $t_{0.975} = 2.228$

Note that as $n$ increases, $t_p$ decreases (and $p > 0.5$).
If you need to get quantiles for $p < 0.5$, use the fact that

$$t_{1-p} = -t_p$$

due to the symmetry of the $t$ distribution.

For example

$$t_{0.4} = -t_{0.6}$$

Also the last row of the table $n = \infty$ corresponds to the quantiles of the $N(0, 1)$ distribution, so instead of inverting the standard normal CDF table, the t-table can be used for some $p$. 

Uses of $t_n, \chi^2_n, F_{m,n}$
For the $\chi^2_n$ distribution, the quantile functions are tabled for selected $p$ and $n$.

$$P[X \leq \chi^2_p] = p$$

When $n$ increases, $\chi^2_p$ increases. This is not surprising, as this corresponds to adding more positive terms to get the distribution.

The table gives quantiles for small and large $p$ since the $\chi^2_n$ distributions are skewed right.
Table 3: Percentiles of $\chi^2$ distribution

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<th>$\chi^2_{0.005}$</th>
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Uses of $tn, \chi^2_n, F_{m,n}$
The tables for the $F_{n_1,n_2}$ are a bit different. While they still satisfy the general quantile relationship

$$P[F \leq F_p(n_1, n_2)] = p$$

there is a different table for for each $p$. The different entries correspond to different combinations of degrees of freedom.

When $n_1$ increases (with fixed $n_2$), $F_p(n_1, n_2)$ increases. However when $n_2$ increases (with fixed $n_1$), $F_p(n_1, n_2)$ decreases.
Table 5: Percentiles of $F$ distribution - $F_{0.9}(n_1, n_2)$

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Uses of $t_n, \chi^2_n, F_{m,n}$
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Uses of $t_n, \chi^2_n, F_{m,n}$