Joint Discrete Distributions

Statistics 110

Summer 2006

Joint Discrete Distributions

Example: Random distribution of 3 balls into 3 cells (all distinguishable)

Sample space has $3^3 = 27$ points

	{Cell 1	Cell 2	Cell 3}		{Cell 1	Cell 2	Cell 3}
1. {	abc	_	_ }	15. {	_	bc	a }
2. {	_	abc	_ }	16. {	c	_	ab }
3. {	_	_	abc }	17. {	b	_	ac }
4. {	ab	c	_ }	18. {	a	_	bc }
			• • •				
12. {	a	bc	_ }	26. {	c	a	b }
13. {	_	ab	c }	27. {	c	b	a }
14. {	_	ac	b }				

Lets define $X_i = \#$ of balls in cell $i, i = 1, 2, 3$ and $N = \#$ number of occupied cells.

The probability of any event involving 2 discrete RVs X and Y can be computed from their joint PMF

$$p_{X,Y}(x,y) = P[X = x, Y = y]$$

(viewed as a function of x and y, $x \in \mathcal{X}, y \in \mathcal{Y}$.)

In the example, $p_{N,X_1}(n, x_1)$ is given by

n \ x_1	0	1	2	3	$p_N(n)$
1	$2/27$	0	0	$1/27$	$1/9$
2	$6/27$	$6/27$	$6/27$	0	$6/9$
3	0	$6/27$	0	0	$2/9$
$p_{X_1}(x_1)$	$8/27$	$12/27$	$6/27$	$1/27$	1

$p_{X_1,X_2}(x_1, x_2)$ is given by

x_1 / x_2	0	1	2	3	$p_{X_2}(x_2)$
0	1/27	3/27	3/27	1/27	8/27
1	3/27	6/27	3/27	0	12/27
2	3/27	3/27	0	0	6/27
3	1/27	0	0	0	1/27
$p_{X_1}(x_1)$	8/27	12/27	6/27	1/27	1

From the joint PMF, we can compute a number of quantities

- Marginal distributions

$$p_X(x) = P[X = x] = \sum_y p_{X,Y}(x, y)$$

is the (marginal) PMF of X. Similarly for Y.

- Joint CDF

$$F_{X,Y}(x,y) = P[X \le x, Y \le y] = \sum_{x_i \le x, y_j \le y} p_{X,Y}(x_i, y_j)$$

- Conditional distributions

$$p_{X|Y}(x|y) = P[X = x | Y = y] = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

This is the conditional PMF of X given $Y = y$. There is one conditional PMF for each value of Y. There is also the conditional distribution of Y given $X = x$.

For example, $p_{X_1|N}(x_1|n)$ is

n \ x_1	0	1	2	3	
1	2/3	0	0	1/3	$\leftarrow$ PMF of $X|N=1$
2	1/3	1/3	1/3	0	$\leftarrow$ PMF of $X|N=2$
3	0	1	0	0	$\leftarrow$ PMF of $X|N=3$

Similarly, there are 4 conditional PMFs of $N|X=x$.

The cases discussed so far only involve 2 RVs. However you can look at the joint distribution of more than 2 RV. For example, the joint distribution of N, X_1, X_2, X_3 which has PMF

$$p_{N,X_1,X_2,X_3}(n, x_1, x_2, x_3) = P[N = n, X_1 = x_1, X_2 = x_2, X_3 = x_3]$$

From this we can get the joint marginal of N and X_1 by

$$p_{N,X_1}(n, x_1) = \sum_{x_2} \sum_{x_3} p_{N,X_1,X_2,X_3}(n, x_1, x_2, x_3)$$

which gives us the table presented earlier.

We can also look at the conditional distribution of X_2 and X_3 given N and X_1. Its PMF has the form

$$\begin{aligned} p_{X_2,X_3|N,X_1}(x_2, x_3|n, x_1) &= P[X_2 = x_2, X_3 = x_3|N = n, X_1 = x_1] \\ &= \frac{p_{N,X_1,X_2,X_3}(n, x_1, x_2, x_3)}{p_{N,X_1}(n, x_1)} \end{aligned}$$

Independent Discrete Random Variables

Two discrete RVs X and Y are independent if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad \text{for all } x \in \mathcal{X}, y \in \mathcal{Y}$$

This is equivalent to saying that the conditional PMF of $X|Y = y$ is the same PMF for all y, or that the conditional PMF of $Y|X = x$ is the same PMF for all x, i.e

$$p_X(x) = p_{X|Y}(x|y); \qquad p_Y(y) = p_{Y|X}(y|x)$$

Theorem. *X and Y are independent discrete RVs if and only if*

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

for all possible events $A \subset \mathcal{X}$ and $B \subset \mathcal{Y}$.

Proof.

$$
\begin{aligned}
P[X \in A, Y \in B] &= \sum_{x \in A} \sum_{y \in B} p_X(x) p_Y(y) \\
&= \left[\sum_{x \in A} p_X(x) \right] \left[\sum_{y \in B} p_Y(y) \right] \\
&= P[X \in A] P[Y \in B]
\end{aligned}
$$

□

Example: Suppose there two hospitals near downtown Boston (call them M and T). The average number of visits to the emergency room due to heart problems are 10/day and 5/day respectively. If we know that on a certain day there are 12 visits in total, what is the joint distribution of the numbers of visits in the two hospitals.

Let $N = M + T$ where M and T are the number of visits to hospitals M and T. Then $P[N = 12]$ satisfies

$$P[N = 12] = P[M = 12, T = 0] + P[M = 11, T = 1] + \ldots + P[M = 0, T = 12]$$

and

$$P[M = m, T = t | N = 12] = \frac{P[M = m, T = t]}{P[N = 12]}; \qquad m + t = 12$$

However we haven't specified enough information to finish this off. Lets assume that $M \sim Pois(10)$ and $T \sim Pois(5)$ and the M and T and independent RVs.

Lets solve this for general the general case

Suppose X and Y are independent Poissons with parameters λ_1 and λ_2 respectively. What is the conditional distribution of X given $X + Y = n$.

Let $N = X + Y$. We want $p_{X|N}(x|n)$ for $x = 0, 1, \ldots, n$.

First the joint distribution of X and N is

$$\begin{aligned} p_{X,N}(x,n) &= P[X = x, N = n] \\ &= P[X = x, Y = n - x] \\ &= P[X = x]P[Y = n - x] \\ &= e^{-\lambda_1}\frac{\lambda_1^x}{x!} \times e^{-\lambda_2}\frac{\lambda_2^{(n-x)}}{(n-x)!} \end{aligned}$$

giving the marginal distribution for N of

$$
\begin{aligned}
p_N(n) &= \sum_{x=0}^{n} p_{X,N}(x,n) \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{x=0}^{n} \frac{1}{x!(n-x)!} \lambda_1^x \lambda_2^{(n-x)} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!} \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\
&\qquad\qquad \text{where } p = \frac{\lambda_1}{\lambda_1+\lambda_2} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}
\end{aligned}
$$

That is $N \sim Pois(\lambda_1 + \lambda_2)$

Then

$$p_{X|N}(x|n) = \frac{e^{-\lambda_1}\frac{\lambda_1^x}{x!} \times e^{-\lambda_2}\frac{\lambda_2^{(n-x)}}{(n-x)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

That is $X|N = n \sim Bin(n, p)$ where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

The concept of "conditional distribution" is very useful.

1. Even if we have $p_{X,N}(x, n)$ for all x, n, this may not give as clear an understanding of the situation as the conditional distribution $p_{X|N}(x|n)$.

2. Once we have the conditional distribution of $X|N = n$, we can compute any other conditional quantity that is defined through the concept of a RV and its distribution.

   e.g. In the above Poisson example

$$E[X|N = n] = np = n\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\text{Var}(X|N = n) = np(1 - p) = n\frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

   So for the example we started out with (assuming independent Poissons)

$$E[M|N = 12] = np = 12\frac{10}{15} = 8$$

$$\text{Var}(M|N = 12) = np(1 - p) = 12\frac{10}{15}\frac{5}{15} = 2.667$$

As part of the above example, we proved a special case of the following

Lemma. *If X and Y are two discrete RVs and $Z = X + Y$, then the PMF of Z is*

$$p_Z(z) = \sum_x p_{X,Y}(x, z - x)$$
$$= \sum_y p_{X,Y}(z - y, y)$$

Furthermore, if X and Y are independently and identically distributed (iid) with PMF $p(\cdot)$, then

$$p_Z(z) = \sum_x p(x)p(z - x)$$

The sequence $p_Z(z), z = \ldots, -2, -1, 0, 1, 2, \ldots,$ is known as the convolution of the sequence $p(\cdot)$ with itself.

$$p_Z(z) = (p * p)(z)$$

Dependent Discrete Random Variables

Often discrete RVs will not be independent. Their joint distribution can still be determined by use of the general multiplication rule.

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x)p_{Y|X}(y|x) \\ &= p_Y(y)p_{X|Y}(x|y) \end{aligned}$$

So in the emergency room visits example, we did not have to assume that the two hospitals were independent.

Example: Polling success rate

When doing a telephone poll, there are a number of results that can occur. It may happen that nobody answers the phone. Or if they answer the phone, they may refused to participate. A possible model describing this situation is

$$N \sim Bin(M, \pi)$$
$$X|N = n \sim Bin(n, p)$$

where M is the number of phone numbers called, N is the number of phone numbers where somebody answers the phone and X is the number of phone numbers where somebody agrees to participate.

The joint PMF of N and X is

$$p_{N,X}(n, x) = \binom{M}{n} \pi^n (1 - \pi)^{M-n} \binom{n}{x} p^x (1 - p)^{n-x}; \quad 0 \leq x \leq n \leq M$$

This is an example of what is known as a hierarchical model.

It is possible to show that the marginal distribution of X is $Bin(M, \pi p)$.

One approach is to show that

$$
\begin{aligned}
p_X(x) &= \sum_{n=0}^{M} p_{N,X}(n, x) \\
&= \sum_{n=0}^{M} \binom{M}{n} \pi^n (1-\pi)^{M-n} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \binom{M}{x} (\pi p)^x (1-\pi p)^{M-x}
\end{aligned}
$$

An easier approach is the following:

For each of the M phone numbers called, a person a can agree to participate or not (a Bernoulli random variable).

For that to happen, two events must occur

1. The phone is answered (with probability π)

2. Given the phone is answered, somebody agrees to participate (with probability p)

The probability that both events occur is πp

So X is the sum of M independent Bernoulli random variables, each with success probability πp.