Transformations Involving Joint Distributions

Statistics 110

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Transformations Involving Joint Distributions

Want to look at problems like

- If X and Y are iid $N(0, \sigma^2)$, what is the distribution of
  - $Z = X^2 + Y^2 \sim Gamma(1, \frac{1}{\sigma^2})$
  - $U = X/Y \sim C(0,1)$
  - $V = X - Y \sim N(0, 2\sigma^2)$

- What is the joint distribution of $U = X + Y$ and $V = X/Y$ if $X \sim Gamma(\alpha, \lambda)$ and $Y \sim Gamma(\beta, \lambda)$ and X and Y are independent.

Approaches:

1. CDF approach $f_Z(z) = \frac{d}{dz} F_Z(z)$

2. Analogue to $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ (Density transformation)

CDF approach:

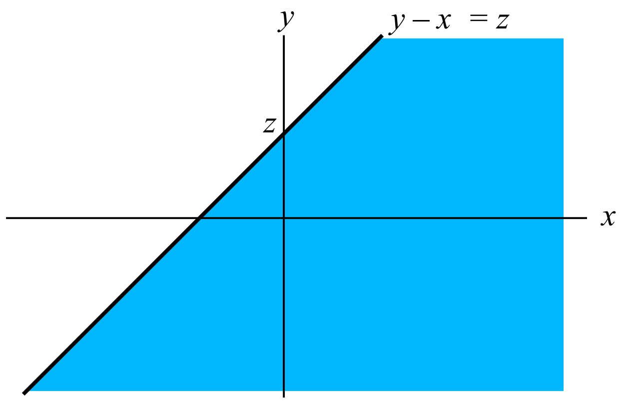
Let $X_1, X_2, \ldots, X_n$ have density $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ and let $Z = g(X_1, X_2, \ldots, X_n)$.

Let $A_z = \{(x_1, x_2, \ldots, x_n) : g(x_1, x_2, \ldots, x_n) \leq z\}$

$$F_Z(z) = P[Z \leq z]$$

Then just differentiate this to get the density

Example: Let $Z = Y - X$. Then $A_z = \{(x, y) : y - x \leq z\} = \{(x, y) : y \leq x + z\}$



$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{x+z} f_{X,Y}(x, y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{y-z}^{\infty} f_{X,Y}(x, y) dx dy \end{aligned}$$

Making the change of variables $x = y - u$ in the second form gives

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z} f_{X,Y}(y-u, y) du dy$$
$$= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_{X,Y}(y-u, y) dy du$$

Differentiating this gives the result

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, x+z) dx$$

by the change of variables $x = y - z$, the alternative form is derived

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(y-z, y) dy$$

For example, let X and Y be independent $N(0,1)$ variables. Then the density of $Z = Y - X$ is

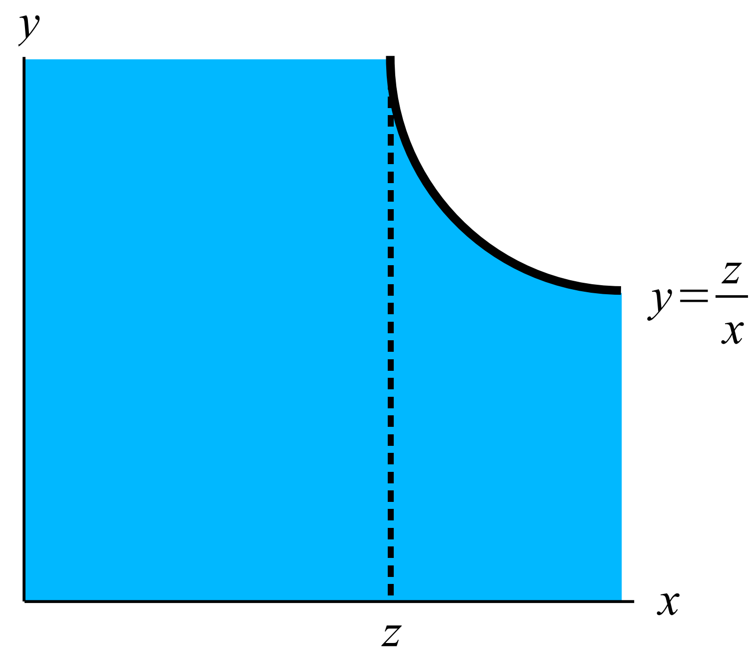
$$
\begin{aligned}
f_Z(z) &= \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+z)^2}{2}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{2x^2 + 2xz + z^2}{2}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{2(x+\frac{z}{2})^2 + \frac{z^2}{2}}{2}\right) dx \\
&= \frac{1}{2\sqrt{2\pi}} e^{-z^2/4} \int_{\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{2(x+\frac{z}{2})^2}{2}\right) dx \\
&= \frac{1}{2\sqrt{2\pi}} e^{-z^2/4}
\end{aligned}
$$

so $Z \sim N(0,2)$

Example: Let $X \sim \beta(a, 1)$ and $Y \sim \beta(b, 1)$ and $Z = XY$ (assume $a > b > 0$).

Then

$$\begin{aligned} A_z &= \{(x, y) : xy \leq z\} \\ &= \{(x, y); 0 \leq x \leq z\} \\ &\quad \cup \{(x, y) : y \leq \frac{z}{x}, z \leq x \leq 1\} \end{aligned}$$



So

$$\begin{aligned} F_Z(z) &= \int_0^z \int_0^1 ax^{a-1} by^{b-1} dy dx + \int_z^1 \int_0^{z/x} ax^{a-1} by^{b-1} dy dx \\ &= \int_0^z ax^{a-1} dx + \int_z^1 ax^{a-1} \left(\frac{z}{x}\right)^b dx \end{aligned}$$

$$F_Z(z) = z^a + az^b \int_z^1 x^{a-1-b} dx$$

$$= z^a + az^b \frac{1}{a-b} z^{a-b} \Big|_z^1$$

$$= z^a + \frac{a}{a-b} z^b (1 - z^{a-b})$$

$$= \frac{a}{a-b} z^b - \frac{b}{a-b} z^a$$

So

$$f_Z(z) = \frac{ab}{a-b} z^{b-1} - \frac{ab}{a-b} z^{a-1} = \frac{ab}{a-b} (z^{b-1} - z^{a-1})$$

Density transformation:

Let X and Y have joint PDF $f_{X,Y}(x,y)$ and suppose

$$U = g_1(X,Y)$$
$$V = g_2(X,Y)$$

is an invertible, differentiable transformation. Assume that the inverse transformation is

$$X = h_1(U,V)$$
$$Y = h_2(U,V)$$

Then the joint density of U and V is

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \left| J_g(x,y) \right|^{-1}$$

where $(x,y) = h(u,v)$ and $J_g(x,y)$ denotes the Jacobian of the function $g(x,y)$

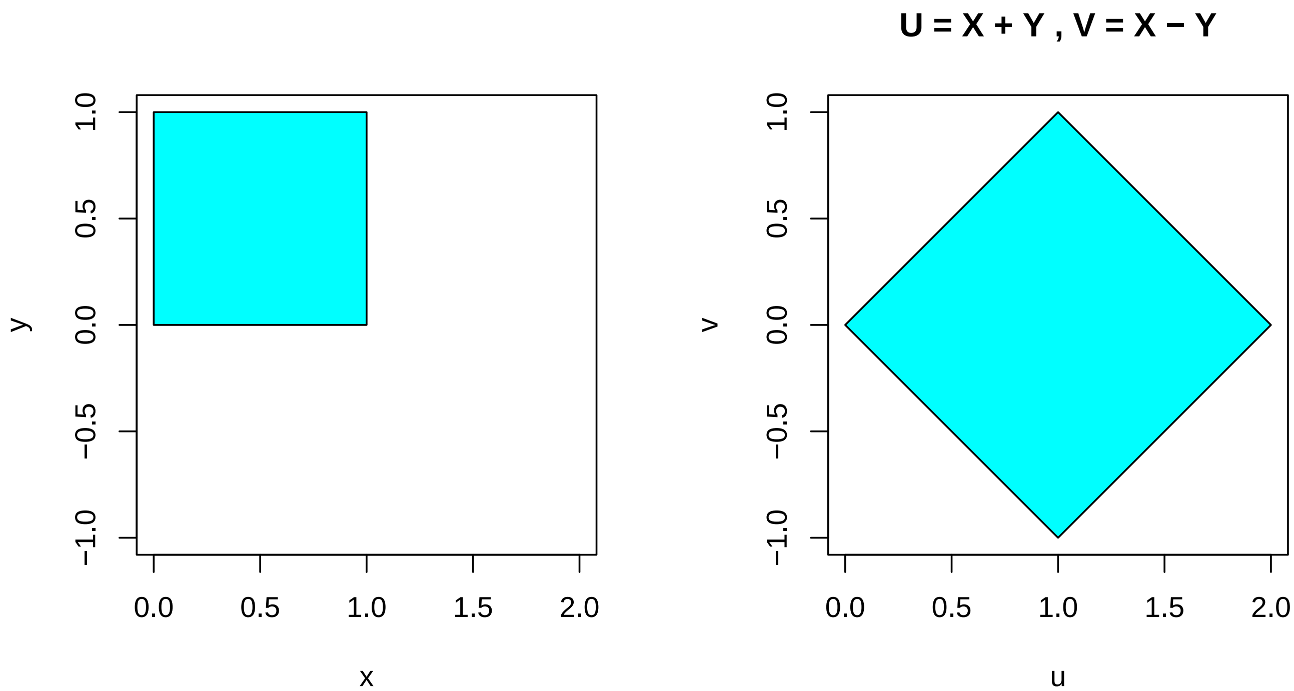
$$J_g = \det \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \frac{\partial g_1}{\partial x}\frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}$$

Like the book, I will not prove this. The idea behind the proof is that when you transform small regions from the (X,Y) space to the (U,V) space the size of the regions changes. The Jacobian gives the multiplicative factor of the size change and what is required for the regions to have the same probabilities in both spaces.

$$U = g_1(X,Y) = X + Y \qquad V = g_2(X,Y) = X - Y$$

$$X = h_1(U,V) = \frac{U+V}{2} \qquad Y = h_2(U,V) = \frac{U-V}{2}$$

$$|J_g| = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = |-2| = 2$$



Lets assume X and Y are iid $U(0,1)$ RVs. Then the joint density of $U = X + Y$ and $V = X - Y$ is

$$f_{U,Y}(u,v) = \frac{1}{2} I_{(0,1)}\left(\frac{u+v}{2}\right) I_{(0,1)}\left(\frac{u-v}{2}\right)$$

So U and V are uniform on the diamond in the previous plot.

Example: Let $X \sim Gamma(a,\lambda)$ be independent of $Y \sim Gamma(b,\lambda)$. What is the joint distribution of $U = X + Y$ and $V = X/Y$.

$$x = h_1(u,v) = \frac{uv}{1+v} \quad y = h_2(u,v) = \frac{u}{1+v}$$

$$|J_g| = \det\begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix} = \left|\frac{-y-x}{y^2}\right| = \frac{u}{(1+v)^2}$$

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{u}{(1+v)^2}\frac{1}{\Gamma(a)\Gamma(b)}\lambda^{a+b}\left(\frac{uv}{1+v}\right)^{a-1}\left(\frac{u}{1+v}\right)^{b-1}e^{-\lambda u} \\
&= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)}u^{a+b-1}e^{-\lambda u}\frac{v^{a-1}}{(1+v)^{a+b}} \\
&= \left\{\frac{\lambda^{a+b}}{\Gamma(a+b)}u^{a+b-1}e^{-\lambda u}\right\}\left\{\frac{1}{\beta(a,b)}\frac{v^{a-1}}{(1+v)^{a+b}}\right\} \\
&= f_U(u)f_V(v)
\end{aligned}
$$

Since the density factors we can see that U and V are independent in this case. In addition $U \sim Gamma(a+b,\lambda)$.

If $b \leq 1$, V has a density with an infinite mean. If $1 < b \leq 2$, V has a finite mean but an infinite variance.

This approach can be generalized to n variables. If $X_1, \ldots, X_n$ has joint PDF $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ and an invertible, differentiable transformation

$$Y_i = g_i(X_1, \ldots, X_n); \quad i = 1, \ldots, n$$

$$X_i = h_i(Y_1, \ldots, Y_n); \quad i = 1, \ldots, n$$

has Jacobian J_g (J_g is the determinant of the matrix with ij entry $\partial g_i / \partial x_j$), then the joint PDF of $Y_1, \ldots, Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) |J_g(x_1, \ldots, x_n)|^{-1}$$

where each of the x_i's is expressed in terms of the y's (e.g. $x_i = h_i(y_1, \ldots, y_n)$

Note that to use this theorem you need as many Y_i's as X_i as the determinant is only defined for square matrices.

If there are less Y_i's than X_i's, (say 1 less), you can set $Y_n = X_n$, apply the theorem, and then integrate out Y_n.

If there are more Y_i's than X_i's, the transformation usually can't be invertible (over determined system), so the theorem can't be applied.