Stat 221: Statistical Computing Methods

Spring, 2004

Solution keys of Assignment 3 Due on Apr. 9, 2004

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1. The given inequality will be rewritten as

$$\begin{aligned} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} &\le \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{1}{\alpha} \alpha x^{\alpha-1} + \frac{1}{\beta} \beta (1-x)^{\beta-1} \right) \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\alpha+\beta}{\alpha\beta} \left(\frac{\beta}{\alpha+\beta} g_1(x) + \frac{\alpha}{\alpha+\beta} g_2(x) \right) \end{aligned}$$

where $g_1(x) = \alpha x^{\alpha-1}$ and $g_2(x) = \beta(1-x)^{\beta-1}$. That is, the proposing distribution is the mixture density and mass constant $M = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\alpha+\beta}{\alpha\beta}$. As explained in my section, the acceptance-rejection method is thus given by

i. Generate x from $g(x)$.

   (a) Generate u from Unif$(0,1)$. If $u \le \beta/(\alpha+\beta)$, let $i = 1$. Otherwise, let $i = 2$.

   (b) Generate x from $g_i(x)$ using the inverse cdf method.

ii. Generate u from Unif$(0,1)$. If $u \le f(x)/(Mg(x))$, accept x. Otherwise, go back to step 1.

Your answer should explain full details of this algorithm.

2.

(a) The crude MC estimator is given by

$$\hat{\theta}_{\text{MC}} = \frac{1}{L} \sum_{\ell=1}^{L} \frac{e^{x^{(\ell)}} - 1}{e - 1}$$

with L draws of x from Unif$(0,1)$.

(b) The importance sampling estimator is given by

$$\hat{\theta}_{\text{IS}} = \frac{1}{L} \sum_{\ell=1}^{L} \frac{e^{x^{(\ell)}} - 1}{e^{x^{(\ell)}}} = \frac{1}{L} \sum_{\ell=1}^{L} 1 - e^{-x^{(\ell)}}$$

with L draws of x from $g(x) = e^x/(e-1)$. The inverse cdf method can be used to get draws from $g(x)$.

(c) We should find the optimal choice of control variates close to the integrand function. Using Taylor expansion $e^x - 1 \approx x$, the control variates estimator is given by

$$\hat{\theta}_{\text{CV}} = \frac{1}{L} \sum_{\ell=1}^{L} \frac{e^{x^{(\ell)}} - 1 - x^{(\ell)}}{e - 1} + \frac{1}{2(e-1)}$$

with L draws of x from Unif$(0,1)$.

(d) Because u and $1-u$ are antithetic each other when $u \sim \text{Unif}(0,1)$, the antithetic variates estimator is given by

$$\hat{\theta}_{\text{AV}} = \frac{1}{L}\left(\sum_{\ell=1}^{L/2} \frac{e^{x^{(\ell)}}-1}{e-1} + \sum_{\ell=L/2+1}^{L} \frac{e^{1-x^{(\ell)}}-1}{e-1}\right)$$

with L draws of x from $\text{Unif}(0,1)$.

Then the simulation results will be easily obtained.

3. The control variates estimator for the variance of the sample median can be obtained by

$$\begin{aligned}
\text{Var}(X_{(n)}) &= \text{E}(X_{(n)})^2 \;=\; \text{E}(X_{(n)} - \bar{X} + \bar{X})^2 \\
&= \text{E}\left((X_{(n)} - \bar{X})^2 + 2\bar{X}(X_{(n)} - \bar{X})\right) + \text{E}(\bar{X})^2 \\
&\approx \frac{1}{L}\sum_{\ell=1}^{L}\left(X_{(n)}^{(\ell)} - \bar{X}^{(\ell)}\right)\left(X_{(n)}^{(\ell)} + \bar{X}^{(\ell)}\right) + \frac{1}{2n-1}.
\end{aligned}$$

4. As shown in my section, the minimization problem of $\text{Var}(\tilde{p}_b)$ hinges on minimizing

$$\begin{aligned}
\text{E}_g\left(h(x)\frac{f(x)}{g(x)}\right)^2 &= \text{E}_g\left(I_{(z\geq c)}\frac{\phi(z)}{\phi(z-b)}\right)^2 \\
&= \int I_{(z\geq c)}\frac{\phi(z)^2}{\phi(z-b)}dz \\
&= \int I_{(z\geq c)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z+b)^2+b^2}dz \\
&= e^{b^2}(1-\Phi(z+b)).
\end{aligned}$$

Solving the first order condition, we have

$$\frac{S(b+c)}{\phi(b+c)} = \frac{1}{2b}$$

and the Mills' ratio implies

$$\frac{1}{b+c}\left(1-\frac{1}{(b+c)^2}\right) \leq \frac{1}{2b} \leq \frac{1}{b+c}.$$

That is, the conditions for the optimal b^* are $b \geq c$ and $b^3 + cb^2 - (c^2+2)b - c^3 \leq 0$. However, using the bisection method, we can find the optimal value of b for $c = 2, 3,$ and 4. Then the corresponding efficiency values can be easily computed.

c	b^*	efficiency
2	2.215930	19.0016
3	3.154849	221.9165
4	4.087179	6350.505